# The Energetic Implications of the Time Discretisation in Implementations of the A.L.E. Equations

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**Abstract**

*A class of A.L.E. time discretisations which inherit key energetic properties (nonlinear dissipation in the absence of forcing and long–term stability under conditions of time dependent loading), irrespective of the time increment employed, is established in this work. These properties are intrinsic to real flows and the conventional Navier–Stokes equations.*

*A description of an incompressible, Newtonian fluid, which reconciles the differences between the various schools of A.L.E. thought in the literature is derived for the purposes of this investigation. The issue of whether these equations automatically inherit the afore mentioned energetic properties must first be resolved. In this way natural notions of nonlinear, exponential–type dissipation in the absence of forcing and long–term stability under conditions of time dependent loading are also formulated.*

*The findings of this analysis have profound consequences for the use of certain classes of finite difference schemes in the context of deforming references. It is significant that many algorithms presently in use do not automatically inherit the fundamental qualitative features of the dynamics.*

*The main conclusions are drawn on in the simulation of a driven cavity flow, a driven cavity flow with various, included rigid bodies, a die–swell problem, and a Stokes second order wave. The improved, second order accuracy of a new scheme for the linearised approximation of the convective term is proved for the purposes of these simulations. A somewhat novel method to generate finite element meshes automatically about included rigid bodies, and which involves finite element mappings, is also described.*

Keywords: Energy conservation; incompressible, newtonian fluid; completely general reference description; arbitrary Lagrangian Eulerian; A.L.E.; free surface; finite elements; new Poincaré inequality; second order accurate linearisation of the convective term; automatic mesh generation.

# 1 Introduction

This work focusses on establishing a class of A.L.E. time discretisations which inherit key energetic properties (nonlinear, exponential–type dissipation in the absence of forcing and long–term stability under conditions of time dependent loading) irrespective of the time increment employed. The findings of this analysis have profound consequences for the use of certain classes of difference schemes in the context of deforming references. It is significant that many algorithms presently in use do not automatically inherit the fundamental qualitative features of the dynamics.

Descriptions of fluid motion are conventionally based on the principles of conservation of mass and linear momentum. One might hope that all such descriptions would accordingly exhibit the afore mentioned, key energetic properties consistant with the principle of energy conservation. These properties are intrinsic to real flows and the conventional, Eulerian Navier–Stokes equations.

A description of an incompressible, Newtonian fluid, which reconciles the differences between the so–called arbitrary Lagrangian Eulerian (A.L.E.) formulation of Hughes, Liu and Zimmerman [6] (deformation gradients absent) and that of Soulaimani, Fortin, Dhatt and Ouellet [15] (deformation gradients present, but use is problematic), is derived for the purposes of this investigation. The implications of the resulting description are investigated in the context of energy conservation in a similar, but broader, approach to that taken by others (eg. Simo and Armero [14]) for the conventional, Eulerian Navier–Stokes equations.

The main conclusions of this work rely on a new inequality and a number of lemmas, the proofs of which are listed in an appendix at the end of the paper. The new inequality is used in place of where the Poincaré–Friedrichs inequality might otherwise have limited the analysis. The lemmas are mainly concerned with the new convective term. This analysis is extended in that non–zero boundaries, so–called free boundaries and time–dependent loads are considered.

The resulting theory is used in the simulation of a driven cavity flow, a driven cavity flow with various, included rigid bodies, a die–swell problem, and a Stokes second order wave. A new scheme for the linearised approximation of the convective term is proposed and the improved, second order accuracy of this scheme is proved for the purposes of these simulations. A somewhat novel method to generate finite element meshes automatically about included rigid bodies, and which involves finite element mappings, is also described.

# 2 A Completely General Reference

The implementation of most numerical time integration schemes would be problematic were a conventional Eulerian[1] description of fluid motion to be used in instances involving deforming domains. The reason is that most numerical time integration schemes require

[1] Eulerian or spatial descriptions are in terms of fields defined over the current configuration.

successive function evaluation at fixed spatial locations (the exception being the finite element with respect to time approach of TEZDUYAR, BEHR and LIOU [18]). On the other hand meshes rapidly snarl when purely Lagrangian[2] descriptions are used. It is for these reasons that a completely general reference description is usually resorted to.

Eulerian and Lagrangian references are just two, specific examples of an unlimited number of configurations over which to define fields used to describe the dynamics of deforming continua. They are both special cases of a more general reference description, a description in which the referential configuration is deformed at will and which is the focus of this investigation. A deforming finite element mesh would be a good example of just such a deforming reference in practice. The transformation to the completely general reference involves coordinates where used as spatial variables only and the resultant description is therefore inertial in the same way as Lagrangian descriptions are.

## 2.1 Domains, Mappings and a Notation

Consider a material body which occupies a domain $\Omega$ at time $t$. The material domain, $\Omega_0$, is that corresponding to time $t = t_0$ (the reference time, $t_0$, is conventionally, but not always, zero). A third configuration, $\tilde{\Omega}$, which is chosen arbitrarily is also defined for the purposes of this work. The three domains are related in the sense that points in one domain may be obtained as one–to–one invertible maps from points in another.

For any general function $f(\boldsymbol{x}, t)$, a function, $\tilde{f}(\tilde{\boldsymbol{x}}, t) \equiv f(\boldsymbol{\lambda}^*(\tilde{\boldsymbol{x}}, t), t)$, can be defined in terms of the domains and one–to–one, invertible mappings illustrated in Fig. 1. Similarly, $f_0(\boldsymbol{x}_0, t) \equiv f(\boldsymbol{\lambda}(\boldsymbol{x}_0, t), t)$ can be defined. This notation can be generalised for the component–wise definition of higher order tensors. The key to understanding much of this work lies possibly in adopting a component-wise defined notation.

In contrast to the function notation just established, the definition of the operators $\tilde{\nabla}$ and $\widetilde{\mathrm{div}}$ is not based on $\nabla$ and div. They are instead the referential counterparts, that is

$$\tilde{\nabla} = \frac{\partial}{\partial \tilde{\boldsymbol{x}}} \qquad \text{and} \qquad \widetilde{\mathrm{div}} = \frac{\partial}{\partial \tilde{x}_1} + \frac{\partial}{\partial \tilde{x}_2} + \frac{\partial}{\partial \tilde{x}_3}.$$

The notation $\boldsymbol{A} : \boldsymbol{B}$ is used to denote the matrix inner product $A_{ij}B_{ij}$ throughout this work, $\langle \cdot \, , \, \cdot \rangle_{L^2(\,\cdot\,)}$ denotes the $L^2$ inner product and $|| \cdot ||_{L^2(\,\cdot\,)}$ the $L^2$ norm.

## 2.2 Some General Results for Functions Defined on the Three Domains

Three important results are necessary for the derivation of the completely general reference description and these are presented below.

[2] LAGRANGIAN or MATERIAL descriptions are made in terms of fields defined over a reference (a material reference) configuration.

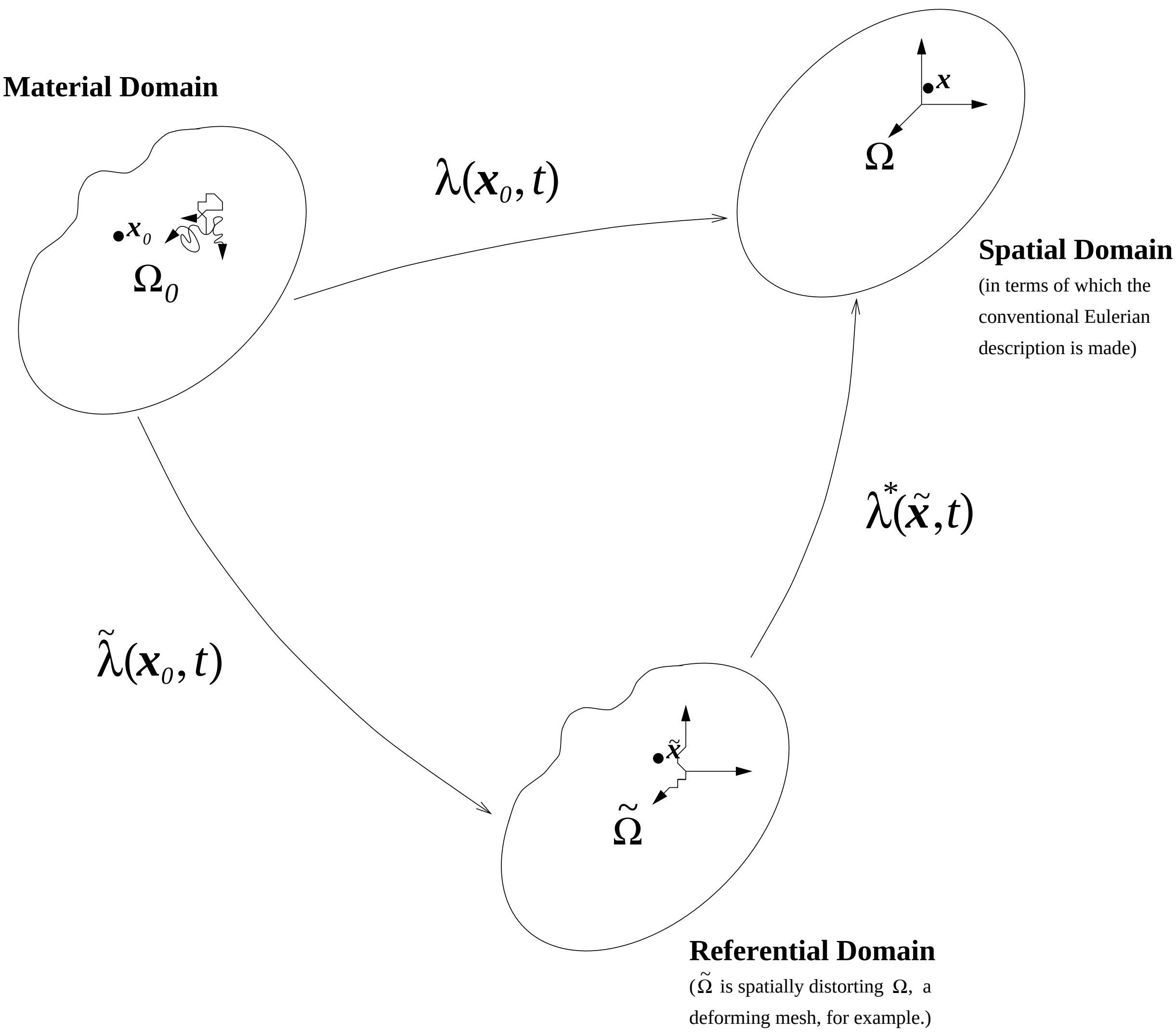



Figure 1: Schematic Diagram of Domains and Mappings Used in a completely general reference Description

### The Material Derivative in Terms of a Completely General Reference

The material derivative of any vector field $\tilde{\boldsymbol{v}}$ in terms of a completely general, reference is

$$\frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + \tilde{\nabla}\tilde{\boldsymbol{v}}\left[\tilde{\boldsymbol{F}}^{-1}(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref})\right]. \tag{1}$$

where $\tilde{\boldsymbol{v}}^{ref}$ is the velocity of the reference deformation, and $\tilde{\boldsymbol{F}}$ is the deformation gradient given by

$$\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}}) = \frac{\partial \boldsymbol{\lambda}^*}{\partial \tilde{\boldsymbol{x}}}.$$

This result is demonstrated in Appendix II.

### An Element of Area in Terms of a Distorting Reference

The second important result can be recalled from general continuum mechanics. Consider an element of area, size $dA$, with an outward unit normal $\boldsymbol{n}$. Then

$$\boldsymbol{n} dA = \tilde{\boldsymbol{F}}^{-t} \tilde{\boldsymbol{N}} \tilde{J} d\tilde{A} \tag{2}$$

where $d\tilde{A}$ and $\tilde{\boldsymbol{N}}$ denote the respective analogous size and outward unit normal of this element of area in the referential configuration and $\tilde{J} = \det \tilde{\boldsymbol{F}}$. This result is demonstrated in most popular textbooks on continuum mechanics (eg. Lai, Rubin and Krempl [10]).

Remark: Notice that $\tilde{\boldsymbol{N}}$ is the single exception to the ˜ quantities devised in this work. $\tilde{\boldsymbol{N}}$ is the surface normal perceived in a completely general reference and the components of $\boldsymbol{n}$ and $\tilde{\boldsymbol{N}}$ need have nothing in common ($\tilde{\boldsymbol{N}}$ is not $\tilde{\boldsymbol{n}}$).

### The Kinematic Relation $\dot{\mathcal{J}}_0 = \mathcal{J}_0 \operatorname{div} \boldsymbol{v}$

The material derivative of the Jacobian $\mathcal{J}_0$ is given by the relation

$$\dot{\mathcal{J}}_0 = \mathcal{J}_0 \operatorname{div} \boldsymbol{v}$$

where $\mathcal{J}_0$ is defined as follows,

$$\mathcal{J}_0 \equiv \det \left\{ \frac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{x}_0} \right\}.$$

This result is demonstrated in most popular textbooks on continuum mechanics (eg. Marsden and Hughes [11]).

## 2.3 Derivation of the Completely General Equation

One way in which to derive a completely general reference description of an incompressible, Newtonian fluid is to start with the balance laws in global (integral) form, and to make the necessary substitutions in these integrals. The desired numerical implementation (similar to the conventional Navier–Stokes one which has been thoroughly investigated and found to be stable) is then obtained.

### Conservation of Mass

Let $\Omega(t)$ be an arbitrary sub–volume of material. The principle of conservation of mass states that

$$\frac{d}{dt} \int_{\Omega(t)} \rho d\Omega = 0 \qquad \textit{(rate of change of mass with time = 0)}$$

$$\frac{d}{dt} \int_{\Omega_0} \rho_0 \mathcal{J}_0 d\Omega_0 = 0 \qquad \textit{(reformulating in terms of the material}$$

*configuration, $\Omega_0$.)*

$$\int_{\Omega_0} \frac{\partial}{\partial t}\{\rho_0 \mathcal{J}_0\}\, d\Omega_0 \;=\; 0 \qquad \textit{(since limits are not time dependent in the material configuration.)} \tag{3}$$

$$\int_{\Omega_0} \left(\rho_0 \dot{\mathcal{J}}_0 \;+\; \dot{\rho}_0 \mathcal{J}_0\right) d\Omega_0 \;=\; 0 \qquad \textit{(by the chain rule)}$$

$$\int_{\Omega(t)} (\dot{\rho} \;+\; \rho\, \mathrm{div}\, \boldsymbol{v})\, d\Omega \;=\; 0 \qquad \textit{(using the kinematic result } \dot{\mathcal{J}}_0 = \mathcal{J}_0 \mathrm{div}\, \boldsymbol{v}\textit{)}$$

$$\int_{\tilde{\Omega}(t)} \left(\dot{\rho} \;+\; \rho \frac{\partial \tilde{v}_i}{\partial \tilde{x}_j}\frac{\partial \tilde{x}_j}{\partial x_i}\right) \tilde{J} d\tilde{\Omega} \;=\; 0 \qquad \textit{(reformulating in terms of the distorting referential configuration, } \tilde{\Omega}(t)\textit{.)}$$

$$\Rightarrow \left(\dot{\rho} \;+\; \rho \tilde{\nabla} \tilde{\boldsymbol{v}} : \tilde{\boldsymbol{F}}^{-t}\right) \tilde{J} \;=\; 0 \qquad \textit{(integrand must be zero since the volume was arbitrary.)}$$

Thus, for a material of constant, non–zero density,

$$\tilde{\nabla} \tilde{\boldsymbol{v}} : \tilde{\boldsymbol{F}}^{-t} = 0 \qquad \text{since} \qquad \tilde{J} \neq 0 \qquad \textit{(mappings are one-to-one and invertible).}$$

Notice also that equation (3) implies

$$\frac{\partial}{\partial t}\{\rho_0 \mathcal{J}_0\} \;=\; 0 \tag{4}$$

since the volume was arbitrary and the integrand must therefore be zero.

## Conservation of Linear Momentum (and Mass)

The principle of conservation of linear momentum for an arbitrary volume of material $\Omega(t)$ with boundary $\Gamma(t)$ states that

$$\frac{d}{dt}\int_{\Omega(t)} \rho \boldsymbol{v} d\Omega = \int_{\Omega(t)} \rho \boldsymbol{b} d\Omega \;+\; \int_{\Gamma(t)} \boldsymbol{\sigma} \boldsymbol{n} dA \tag{5}$$

where $\rho$ is density, $\boldsymbol{b}$ is the body force per unit mass, $\boldsymbol{\sigma}$ is the stress, $\boldsymbol{n}$ the outward unit normal to the boundary and $\boldsymbol{v}$ is the velocity. The term on the lefthand side can be rewritten as follows:

$$\frac{d}{dt}\int_{\Omega(t)} \rho \boldsymbol{v} d\Omega \;=\; \frac{d}{dt}\int_{\Omega_0} \rho_0 \boldsymbol{v}_0 \mathcal{J}_0 d\Omega_0 \qquad \textit{(Reformulating in terms of the material configuration, } \Omega_0\textit{.)}$$

$$= \int_{\Omega_0} \frac{\partial}{\partial t}\{\rho_0 \boldsymbol{v}_0 \mathcal{J}_0\}\, d\Omega_0 \qquad \textit{(Since limits are not time dependent in the material configuration.)}$$

$$= \int_{\Omega_0} \left(\frac{\partial \boldsymbol{v}_0}{\partial t} \rho_0 \mathcal{J}_0 \;+\; \boldsymbol{v}_0 \frac{\partial}{\partial t}\{\rho_0 \mathcal{J}_0\}\right) d\Omega_0$$

$$
\begin{aligned}
&= \int_{\Omega(t)} \rho \dot{\boldsymbol{v}} d\Omega && \textit{(The second term above is zero as a consequence of equation (4).)} \\
&= \int_{\tilde{\Omega}(t)} \rho \dot{\tilde{\boldsymbol{v}}} \tilde{J} d\tilde{\Omega} && \textit{(Reformulating in terms of the dist–orting referential configuration, } \tilde{\Omega}.\textit{)} \\
&= \int_{\tilde{\Omega}(t)} \rho \left( \frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + \tilde{\nabla} \tilde{\boldsymbol{v}} \left[ \tilde{\boldsymbol{F}}^{-1} (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}) \right] \right) \tilde{J} d\tilde{\Omega} && \textit{(Using result (1) on page 4)}
\end{aligned}
$$

where $\dot{\boldsymbol{v}}$ denotes the material derivative of $\boldsymbol{v}$. The surface integral becomes

$$
\begin{aligned}
\int_{\Gamma(t)} \boldsymbol{\sigma} \boldsymbol{n} dA &= \int_{\tilde{\Gamma}(t)} \tilde{\boldsymbol{\sigma}} \tilde{\boldsymbol{F}}^{-t} \tilde{\boldsymbol{N}} \tilde{J} d\tilde{A} && \textit{(Reformulating in terms of a distorting reference using result (2) on page 5.)} \\
&= \int_{\tilde{\Omega}(t)} \widetilde{\operatorname{div}} \, \{ \tilde{\boldsymbol{\sigma}} \tilde{\boldsymbol{F}}^{-t} \tilde{J} \} d\tilde{\Omega} && \textit{(By the divergence theorem).}
\end{aligned}
$$

Finally, the term involving body force becomes

$$
\int_{\Omega(t)} \rho \boldsymbol{b} d\Omega = \int_{\tilde{\Omega}(t)} \rho \tilde{\boldsymbol{b}} \tilde{J} d\tilde{\Omega} \qquad \textit{(Reformulating in terms of a distorting reference.).}
$$

Substituting these expressions into (5), remembering that the volume used in the argument was arbitrary and that the entire integrand must therefore be zero, the conservation principles of linear momentum and mass may be written in primitive form as

$$
\rho \left( \frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + \tilde{\nabla} \tilde{\boldsymbol{v}} \tilde{\boldsymbol{F}}^{-1} (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}) \right) \tilde{J} = \rho \tilde{\boldsymbol{b}} \tilde{J} + \widetilde{\operatorname{div}} \, \tilde{\boldsymbol{P}} \tag{6}
$$

and

$$
\tilde{\nabla} \tilde{\boldsymbol{v}} : \tilde{\boldsymbol{F}}^{-t} = 0 \tag{7}
$$

where $\tilde{\boldsymbol{P}}$ is the Piola–Kirchoff stress tensor of the first kind, $\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{\sigma}} \tilde{\boldsymbol{F}}^{-t} \tilde{J}$. In terms of the constitutive relation, $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu \boldsymbol{D}$, for a Newtonian fluid,

$$
\tilde{\boldsymbol{P}} = \left( -p\boldsymbol{I} + \mu \left[ \tilde{\nabla} \tilde{\boldsymbol{v}} \tilde{\boldsymbol{F}}^{-1} + \left( \tilde{\nabla} \tilde{\boldsymbol{v}} \tilde{\boldsymbol{F}}^{-1} \right)^t \right] \right) \tilde{\boldsymbol{F}}^{-t} \tilde{J} \qquad \text{since} \qquad \tilde{\boldsymbol{D}} = \frac{1}{2} \left( \widetilde{\nabla \boldsymbol{v}} + \left( \widetilde{\nabla \boldsymbol{v}} \right)^t \right).
$$

The derivation of a variational formulation is along similar lines as that for the Navier-Stokes equations (the purely Eulerian description). For a fluid of constant density, the variational formulation

$$
\begin{aligned}
\rho \int_{\tilde{\Omega}} \tilde{\boldsymbol{w}} \cdot \frac{\partial \tilde{\boldsymbol{v}}}{\partial t} \tilde{J} d\tilde{\Omega} \; + \; \rho \int_{\tilde{\Omega}} \tilde{\boldsymbol{w}} \cdot \tilde{\nabla} \tilde{\boldsymbol{v}} \left[ \tilde{\boldsymbol{F}}^{-1} (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}) \right] \tilde{J} d\tilde{\Omega} \; = \\
\rho \int_{\tilde{\Omega}} \tilde{\boldsymbol{w}} \cdot \tilde{\boldsymbol{b}} \tilde{J} d\tilde{\Omega} \; + \; \int_{\tilde{\Omega}} \tilde{p} \tilde{\nabla} \tilde{\boldsymbol{w}} : \tilde{\boldsymbol{F}}^{-t} \tilde{J} d\tilde{\Omega} \; - \; 2\mu \int_{\tilde{\Omega}} \tilde{\boldsymbol{D}}(\tilde{\boldsymbol{w}}) : \tilde{\mathbf{D}}(\tilde{\boldsymbol{v}}) \tilde{J} d\tilde{\Omega} \\
+ \rho \int_{\tilde{\Gamma}} \tilde{\boldsymbol{w}} \tilde{\boldsymbol{P}} \tilde{\boldsymbol{N}} d\tilde{\Gamma}
\end{aligned} \tag{8}
$$

$$\int_{\tilde{\Omega}} \tilde{q} \tilde{\nabla} \tilde{\boldsymbol{v}} : \tilde{\boldsymbol{F}}^{-t} d\tilde{\Omega} \;=\; 0 \tag{9}$$

is obtained, where $\tilde{q}$ and $\tilde{\boldsymbol{w}}$ are respectively the arbitrary pressure and velocity of the variational formulation.

Notice that the usual procedure of assigning a value of zero to the arbitrary velocity, $\tilde{\boldsymbol{w}}$, at the boundary has not been followed. The boundary integral in the variational momentum equation has consequently not been eliminated as is normally done. The reasons are twofold; firstly problems for which the ensuing investigation is intended are of a free boundary type and so the solution is not known there; secondly, a specific function (which cannot arbitrarily be assigned a value of zero at the boundary) will be substituted for $\tilde{\boldsymbol{w}}$ in the forthcoming analysis.

## 2.4 Reconciling the Different Schools of Thought

The equations (6) and (7) are the completely general referential description of an incompressible, Newtonian fluid. They reduce to the so–called A.L.E. equations of Hughes, Liu and Zimmerman [6] for an instant in which spatial and referential configurations coincide.

Since the approximate set of equations is broken into a sequence of discrete time steps in the implementation, one is entitled to choose a new referential configuration during each time step, should one so desire. This is what is known as an "updated" approach; when each time step is really a fresh implementation. In the case of time stepping schemes based about a single instant (eg. the generalised class of Euler difference schemes to be investigated in Section 4) a considerably simplified implementation can be achieved by an appropriate choice of configurations. Making the choice of a referential configuration which coincides with the spatial configuration at the instant about which the time stepping scheme is based allows the deformation gradient to be omitted from the approximation altogether (the deformation gradient is identity under such circumstances). For such implementations (those which require evaluation about a single point only) no error arises from the use of the equations cited in Hughes, Liu and Zimmerman [6],

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \nabla \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{v}^{ref})\right) \;=\; \rho \boldsymbol{b} + \operatorname{div} \boldsymbol{\sigma} \tag{10}$$

$$\operatorname{div} \boldsymbol{v} \;=\; 0. \tag{11}$$

These equations are not valid for any, arbitrary choice of reference or if the implementation requires the equation to be evaluated at more than one point within each time step (eg. a Runge–Kutta or finite–element–in–time scheme). It is important to remember that in a discrete context the reference configuration is fixed for the duration of the entire time increment. Although the referential configuration is hypothetical and can be chosen arbitrarily for each time step, once chosen it is static for the duration of the entire time step. Once the coincidence of configurations is ordained at a given instant, $\tilde{\boldsymbol{F}}$ is defined by the reference (mesh) deformation, both before and after, and must be consistant.

The equations of Hughes et al. are an arbitrary Lagrangian Eulerian (A.L.E.) description in the very true sense under the circumstances of implementations requiring

evaluation about more than one point within each time step (this is not surprising considering the equations have their origins in the arbitrarily, either Lagrangian or Eulerian programmes of HIRT, AMSDEN and COOK [5]). This fact is further bourne out in observing that key energetic properties, consistant with the principle of energy conservation, are not automatically inherited by the equations of HUGHES ET. AL. in the context of more general references.

The momentum equations of SOULAIMANI, FORTIN, DHATT and OUELLET [15] are flawed as a result of the mistaken belief that $\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{F}}^{-1}\tilde{J}$ is the Piola–Kirchoff stress tensor of the first kind (pg. 268 of SOULAIMANI ET AL.). Yet another problem is illustrated by rewriting the conventional incompressibility condition using the chain rule. The new incompressibility condition which arises is most certainly

$$\frac{\partial \tilde{v}_i}{\partial \tilde{x}_j}\frac{\partial \tilde{x}_j}{\partial x_i} = 0 \qquad \text{and not} \qquad \frac{\partial \tilde{v}_i}{\partial \tilde{x}_j}\frac{\partial \tilde{x}_i}{\partial x_j} = 0.$$

Further errors arising (eg. $\hat{J}$ omitted in the first term on the right hand side of the momentum equation, equation (10) on pg. 268 of SOULAIMANI ET AL.) make the use of these equations problematic.

There would seem to be no reason why one would wish to define the deformation about a configuration other than that at the instant about which the implementation is based (assuming the implementation used is indeed based about a single point eg. a finite difference) thereby involving deformation gradients. Resolving the resulting difficulties associated with the deformation gradients by means of a perturbation seems unnecessarily complicated in the light of the above reasoning.

# 3 Natural Notions of Energy Conservation in Terms of the Completely General Equations

The effect of quantities parameterising reference deformation on key energetic properties – nonlinear, exponential–type dissipation in the absence of forcing and long–term stability under conditions of time dependent loading – is investigated in this section. These properties, of form

$$K(\boldsymbol{v}) \leq K(\boldsymbol{v}\mid_{t_0})\; e^{-2\nu C t} \qquad \text{and} \qquad \lim_{t\to\infty} \sup K(\boldsymbol{v}) \leq \frac{M^2}{2\nu^2 C^2}$$

respectively (where $K = \frac{1}{2}\rho\,||\boldsymbol{v}||^2_{L^2(\Omega)}$ is the total kinetic energy), are intrinsic to real flows and the conventional, Eulerian Navier–Stokes equations (see TEMAM [16], [17], CONSTANTIN and FOIAS [2] and SIMO and ARMERO [14] in this regard). The effect of $\tilde{\boldsymbol{v}}^{ref}$ on the afore mentioned aspects of conservation of the quantity

$$\tilde{K}(\tilde{\boldsymbol{v}}) \equiv \frac{1}{2}\rho\left|\left|\tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}$$

is essentially what is being investigated, with a view to establishing a set of conditions under which the discrete approximation can reasonably be expected to inherit these self–same energetic properties.

One might anticipate key energetic properties to be manifest only in instances involving a fixed contributing mass of material, whether its boundaries be dynamic, or not. An analysis of this nature only makes sense in the context of a constant volume of incompressible fluid.

Inequalities of the Poincaré-Friedrichs type are a key feature of any stability analysis of this nature. Gradient containing $L^2$ terms need to be re–expressed in terms of energy. In the case of a "no slip" ($\boldsymbol{v} = 0$) condition on the entire boundary the situation is straightforward, in that it is possible to use the standard Poincare-Friedrichs inequality: there exists a constant $C_1 > 0$ such that

$$\|\boldsymbol{v}\|_{L^2} \leq C_1 \|\nabla \boldsymbol{v}\|_{L^2} \quad \text{for all } \boldsymbol{v} \in [H_0^1(\Omega)]^n.$$

The use of the classical Poincaré–Friedrichs inequality is otherwise identified as a major limitation, even in the conventional Navier–Stokes related analyses. The Poincaré–Friedrichs inequality is only applicable in very limited instances where the value for the entire boundary is stipulated to be identically zero. For boundary conditions of a more general nature, such as those encountered in this study, in which parts of the boundary may be either a free surface, have an imposed velocity or be subject to traction conditions, a more suitable inequality is required. (It should, however, be noted that subtracting a boundary velocity and analysing the resulting equation is nonetheless still a feasable alternative, despite the fact that the equations are nonlinear. This approach requires a more sophisticated and involved level of mathematics[1].)

Some common boundary types and associated descriptions are briefly summarised as follows:

1. **Fixed impermeable boundaries:** The description at such boundaries is usually Eulerian and the quantities $\tilde{\boldsymbol{F}}$ and $\tilde{\boldsymbol{v}}^{ref}$ consequently become identity and zero respectively. These are typically (but not always) "no slip" boundaries, implying that $\tilde{\boldsymbol{v}} \mid_\Gamma = \boldsymbol{0}$.

2. **Free boundaries:** Conventional use allows the spatial mesh to slide along free boundaries while still maintaining their overall Lagrangian character. Stated more formally,
$$\tilde{\boldsymbol{n}} \cdot \left( \tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref} \right) = 0.$$
The total volume is nonetheless still a material volume overall.

3. **Imposed velocity-type boundaries:** Conventional use entails descriptions which usually become pure Eulerian at such boundaries. The total flow across such boundaries is zero for an incompressible fluid if volume is to be preserved. For boundary–driven flows one therefore usually assumes that the quantity
$$\int_\Gamma \boldsymbol{n} \cdot \boldsymbol{v} \; d\Gamma$$
vanishes (á la TEMAM [17]).

[1] TEMAM [17] succeeds in arriving at an estimate which proves the existance of a maximal attractor in two dimensions in this manner.

4. **Imposed traction-type boundaries:** A variety of descriptions are used at such boundaries, ranging from pure Eulerian to the vanishing $\tilde{\boldsymbol{n}}\cdot\left(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}\right)$ type described for free boundaries.

These are the modes of reference deformation commonly used at boundaries encountered in practice and which will need to be accommodated if the theory is to be applicable.

The particular types of geometry considered are those that arise in problems involving the motion of rigid bodies such as pebbles on the sea bed; thus a free surface is present, and the domain may be multiply connected. The Poincaré–Friedrichs inequality does, furthermore, not hold on subdomains of the domain in question and the constant is not optimal.

Further investigation (COMMUNICATION [13]) reveals a similar relation, the so-called Poincaré–Morrey inequality, holds providing the function attains a value of zero somewhere on the boundary. The proof of the Poincaré-Morrey inequality is, however, similar to that of one of Korn's inequalities (see, for example, KIKUCHI and ODEN [7]). In particluar, it is non–constructive, by contradiction and the constant cannot therefore be determined as part of the proof. Viewed in this light the forthcoming inequality amounts to a specification of the hypothetical constant in the Poincaré–Morrey inequality for domains having a star–shaped geometry.

INEQUALITY 1 (A NEW "POINCARÉ" INEQUALITY) *Suppose $\boldsymbol{v}$ is continuous and differentiable to first order and that $\boldsymbol{v}$ attains a maximum absolute value, c, on an included, finite neighbourhood of minimum radius $R_{\min}$ about a point $\boldsymbol{x}^{\text{origin}}$ (as depicted in Fig. 2).*

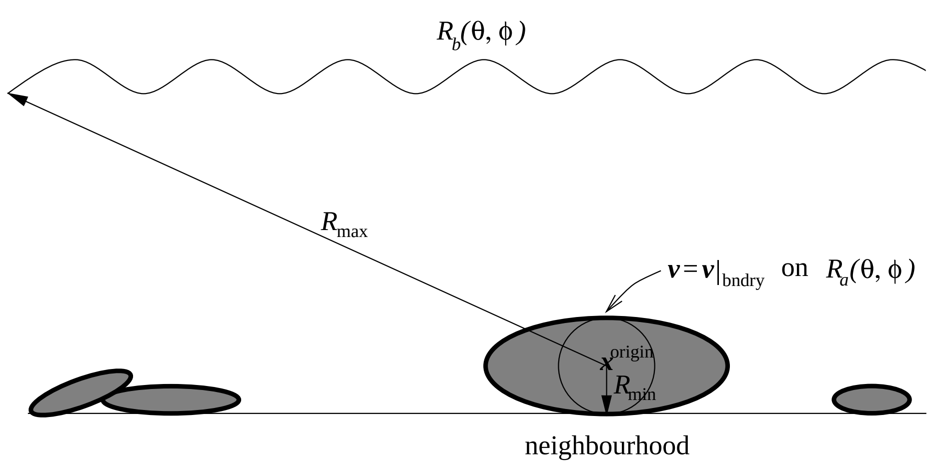



Figure 2: A Finite Neighbourhood of Minimum Radius $R_{\min}$ About a Point $\boldsymbol{x}^{\text{origin}}$.

*If* $\Omega$ *is a bounded, star–shaped (about a point* $\boldsymbol{x}^{\text{origin}}$*)*[1] *domain in* $R^3$*, then*

$$||\boldsymbol{v}||_{L^2(\Omega)} \leq \left[\frac{(R_{\max}-R_{\min})(R_{\max}^3-R_{\min}^3)}{3R_{\max}R_{\min}}\right]^{\frac{1}{2}} ||\nabla\boldsymbol{v}||_{L^2(\Omega)} + ||c||_{L^2(\Omega)}$$

*where* $R_{\max}$ *is the distance to the farthest point in* $\Omega$ *from* $\boldsymbol{x}^{\text{origin}}$*. (Proof in Appendix II.)*

Remark: Notice that "no slip" boundaries which contravene the star–shaped requirement are not of any consequence. This is since additional contributions to inequality terms, arising due to the inclusion of any such intruding domains, can be arbitrarily costructed to have a value of zero (without any loss of generality). If for example, one were to apply the inequality in an investigation of the flow around an aeroplane wing, one might make the convenient choice of the wing interior as the desired neighbourhood.

Remark: Notice that "no slip" boundaries which contravene the star–shaped requirement are not of any consequence. This is since additional contributions to inequality terms, arising due to the inclusion of any such intruding domains, can be arbitrarily costructed to have a value of zero (without any loss of generality). If for example, one were to apply the inequality in an investigation of the flow around an aeroplane wing, one might make the convenient choice of the wing interior as the desired neighbourhood.

This inequality is similar to the Poincaré–Friedrichs inequality when $c = 0$, but is extended to a geometrical subclass of domains which have free and partly non-zero boundaries in addition to being more applicable to more challenging examples such as the flow around an aeroplane wing. It has a further advantage in that the coefficient is an order of magnitude more optimal when used under the "no slip" Poincaré–Friedrichs condition (under such conditions the domain can always be deconstructed into a number of subdomains in which $R_{\min} = \frac{1}{3}R_{\max}$). The Poincaré–Friedrichs inequality is a special case of the above inequality. The necessary lemma (below) follows naturally from Inequality 1.

Lemma 1 (Deviatoric Stress Term Energy) *The kinetic energy satisfies the bound* $\dfrac{C}{\rho}\tilde{K}(\tilde{\boldsymbol{v}}) \;\leq\; \left|\left|\tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}})\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}$*, where* $C$ *is related to the constant in Inequality 1,* $C > 0$*. (Proof in Appendix II.)*

As the reviewers rightly point out, this is Korn's inequality with a specified constant limited to star–shaped geometries and a variety of such inequalities can be found on page 323 of Marsden and Hughes [11]. The following lemma will facilitate the elimination of the convective energy rate in the forthcoming analysis.

Lemma 2 (Convective Energy Rate) *The relation*

$$-\rho\left\langle\tilde{\boldsymbol{v}},(\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} = -\frac{1}{2}\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\left(\tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)\right)\tilde{J}\right\rangle_{L^2(\tilde{\Gamma})} -\frac{1}{2}\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\frac{\partial\tilde{J}}{\partial t}\right\rangle_{L^2(\tilde{\Omega})}$$

[1] by which is meant that every point in the domain can be reached by a straight line from $\boldsymbol{x}^{\text{origin}}$ that does not pass outside of $\Omega$

*holds under the conditions required for equations (10) and (11) to be a completely general reference description. (Proof in Appendix II.)*

The above lemma is crucial to the analysis for deforming references in particular. The following lemma will establish that the boundary term vanishes at free boundaries under conditions of conventional usage.

LEMMA 3 (FREE BOUNDARY ENERGY RATE) *The boundary term*

$$-\frac{1}{2}\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\left(\tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)\right)\tilde{J}\right\rangle_{L^2(\tilde{\Gamma})}$$

*vanishes at free boundaries provided the description there is of a vanishing* $\tilde{\boldsymbol{n}}\cdot\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)$ *type. (Proof in Appendix II.)*

This concludes the preliminaries required for the deforming reference energy analysis.

## 3.1 Exponential Dissipation in the Absence of Forcing

The issue of whether nonlinear, exponential–type dissipation in the absence of forcing is a property intrinsic to the deforming reference description is resolved by the following theorem.

THEOREM 1 (EXPONENTIAL DISSIPATION IN THE ABSENCE OF FORCING) *A sufficient condition for the completely general reference description to inherit nonlinear, exponential type energy dissipation*

$$\tilde{K}(\tilde{\boldsymbol{v}})\leq\tilde{K}(\tilde{\boldsymbol{v}}\mid_{t_0})\;e^{-2\nu Ct}$$

*in the absence of forcing is that the reference moves in a vanishing* $\tilde{\boldsymbol{n}}\cdot\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)$ *fashion at free boundaries and becomes pure Eulerian at boundaries of a fixed, impermeable type (the conventional use).*

PROOF: Notice that an expression involving the kinetic energy can be formulated by substituting $\tilde{\boldsymbol{v}}$ for $\tilde{\boldsymbol{w}}$ in the variational momentum equation (8). Then

$$\begin{aligned}\rho\left\langle\tilde{\boldsymbol{v}},\frac{\partial\tilde{\boldsymbol{v}}}{\partial t}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} &= \left\langle\tilde{p}\tilde{\nabla}\tilde{\boldsymbol{v}},\tilde{\boldsymbol{F}}^{-t}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}-2\mu\left\langle\tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}}),\tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}})\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}\\ &\quad -\rho\left\langle\tilde{\boldsymbol{v}},(\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}\\ &\quad +\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{b}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}+\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right\rangle_{L^2(\tilde{\Gamma})}.\end{aligned}\tag{12}$$

The order of integration and differentiation are fully interchangeable (the volume is still a material volume overall for the type of time–dependent limits associated with free boundaries). The term containing the pressure, that is

$$\left\langle\tilde{p}\tilde{\nabla}\tilde{\boldsymbol{v}}:\tilde{\boldsymbol{F}}^{-t}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})},$$

vanishes as a result of incompressibility (equation (7)). Equation (12) can accordingly be rewritten

$$\begin{aligned}\frac{1}{2}\rho\left(\frac{d}{dt}\left|\left|\tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}-\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\frac{\partial\tilde{J}}{\partial t}\right\rangle_{L^2(\tilde{\Omega})}\right) &= \tilde{K}_{\tilde{\Gamma}_{\text{imposed }\tilde{\boldsymbol{v}}}}-2\mu\left|\left|\tilde{\boldsymbol{D}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}\\ &\quad-\rho\left\langle\tilde{\boldsymbol{v}},(\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}\\ &\quad+\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{b}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}+\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right\rangle_{L^2(\tilde{\Gamma})}\end{aligned}$$

where $\tilde{K}_{\tilde{\Gamma}_{\text{imposed }\tilde{\boldsymbol{v}}}}$ is zero for the present by virtue of the fully interchangeable orders of integration and differentiation (its meaning will be made clear in the pages to follow). Using Lemmas 1 and 2 an expression

$$\begin{aligned}\frac{d\tilde{K}(\tilde{\boldsymbol{v}})}{dt} &\leq -2\nu C\tilde{K}(\tilde{\boldsymbol{v}})+\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{b}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}+\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right\rangle_{L^2(\tilde{\Gamma})}\\ &\quad-\frac{1}{2}\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\left(\tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)\right)\tilde{J}\right\rangle_{L^2(\tilde{\Gamma})}+\tilde{K}_{\tilde{\Gamma}_{\text{imposed }\tilde{\boldsymbol{v}}}}\end{aligned}\tag{13}$$

is obtained, where $\tilde{K}=\frac{1}{2}\rho\left|\left|\tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}$ is the total kinetic energy.

The term $\tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref})$ vanishes at fixed impermeable boundaries since both $\tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{v}}^{ref}$ vanish under such circumstances (assuming the description becomes purely Eulerian there). This self–same term also vanishes at free boundaries according to Lemma 3. Boundaries of an imposed velocity type need not be accounted for as a consequence of the stated “no forcing” condition, and so

$$\frac{d\tilde{K}(\tilde{\boldsymbol{v}})}{dt} \leq -2\nu C\tilde{K}(\tilde{\boldsymbol{v}})+\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{b}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}+\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right\rangle_{L^2(\tilde{\Gamma})}.$$

This equation has a solution of the form

$$\tilde{K}\leq\tilde{K}(\tilde{\boldsymbol{v}}\mid_{t_0})\;e^{-2\nu Ct}$$

in the absence of forcing (*“no forcing”* $\Rightarrow\tilde{\boldsymbol{b}}=\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}=\boldsymbol{0}$).

A nonlinear, exponential–type energy dissipation in the absence of forcing is therefore an intrinsic property of the completely general referential description. This contractive flow property is also an intrinsic property of the conventional Navier–Stokes equations.

## 3.2 Long–Term Stability under Conditions of Time–Dependent Loading

The formulation of suitable load and free surface bounds is necessary before the issue of long-term stability ($L^2$–stability) under conditions of time–dependent loading can be resolved. The energy transfer across boundaries at which there is an imposed velocity is a further factor which must be taken into account under conditions of forcing. The following lemma facilitates the formulation of load and free surface bounds.

Lemma 4 (Force, Free Surface Bounds) *The inequality*

$$\rho\left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{b}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} + \left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right\rangle_{L^2(\tilde{\Gamma})} \leq \frac{\nu C}{2}\left(\rho\left|\left|\tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})} + \left|\left|\tilde{\boldsymbol{v}}\right|\right|^2_{L^2(\tilde{\Gamma})}\right) + \frac{1}{2\nu C}\left(\rho\left|\left|\tilde{\boldsymbol{b}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})} + \left|\left|\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right|\right|^2_{L^2(\tilde{\Gamma})}\right)$$

*holds where $\nu C$ is a constant, $\nu C > 0$. (Proof in Appendix II.)*

The relation immediately below will negate any convection–related contribution to the energy bound at imposed velocity–type boundaries.

Lemma 5 (Convective Energy Rate at an Imposed Velocity–Type Boundary) *The relation*

$$-\frac{1}{2}\rho\left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}\left(\tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot\left(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}\right)\right)\tilde{J}\right\rangle_{L^2(\tilde{\Gamma})} \leq 0$$

*holds for boundaries at which there is an imposed velocity provided there is no nett inflow/outflow across such boundaries and the description there becomes pure Eulerian. (Proof in Appendix II.)*

This done, the mathematical machinery necessary to the long–term stability analysis is in place.

Theorem 2 (Long–Term Stability) *A sufficient condition for the completely general reference description to inherit the property of long–term stability*

$$\lim_{t\to\infty}\sup \tilde{K}(\tilde{\boldsymbol{v}}) \leq \frac{M^2}{2\nu^2C^2}$$

*under conditions of time–dependent loading, where this time–dependent loading, the speed of the surface and any imposed boundary velocity is bounded in such a way that*

$$\rho\left|\left|\tilde{\boldsymbol{b}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})} + \left|\left|\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right|\right|^2_{L^2(\tilde{\Gamma})} + \nu^2C^2\left|\left|\tilde{\boldsymbol{v}}\right|\right|^2_{L^2(\tilde{\Gamma})} + K^2_{\tilde{\Gamma}\text{imposed } \tilde{\boldsymbol{v}}} \leq M^2, \qquad \dagger$$

*is that the description is of a vanishing $\tilde{\boldsymbol{n}}\cdot\left(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}\right)$ type at free boundaries, that it becomes purely Eulerian at boundaries across which there is an imposed velocity or where boundaries are of a fixed, impermeable type.*

Proof: Cognizance must now be taken of a previously unencountered boundary type; that of a stationary boundary across which there is an imposed velocity. The limits of the integral on the left hand side of equation (12) are time–dependant under such circumstances and the volume is no longer a material volume overall. $\tilde{K}_{\tilde{\Gamma}\text{imposed } \tilde{\boldsymbol{v}}}$ in equation

† The additional terms $\tilde{K}_{\tilde{\Gamma}\text{imposed } \tilde{\boldsymbol{v}}}$ are given in Appendix I. They are only applicable in instances where there is an imposed velocity at the boundary. The other two boundary terms are only applicable at boundaries which involve tractions.

(13) is no longer zero. Using Lemmas 3, 4 and 5 in equation (13), then applying the above bound,

$$\frac{d\tilde{K}(\tilde{\boldsymbol{v}})}{dt} + \nu C \tilde{K}(\tilde{\boldsymbol{v}}) \;\; \leq \;\; \frac{M^2}{2\nu C},$$

which, when solved, yields

$$\tilde{K}(\tilde{\boldsymbol{v}}) \;\; \leq \;\; e^{-\nu C t}\tilde{K}(\tilde{\boldsymbol{v}}\mid_{t=t_0}) + \left(1 - e^{-\nu C t}\right)\frac{M^2}{2\nu^2 C^2}.$$

This in turn implies

$$\lim_{t\to\infty}\sup \tilde{K}(\tilde{\boldsymbol{v}}) \;\; \leq \;\; \frac{M^2}{2\nu^2 C^2}.$$

The preceding analyses lead to natural notions of nonlinear dissipation in the absence of forcing and long–term stability under conditions of time–dependent loading for the analytic problem. These properties are also intrinsic features of real flows and the Navier–Stokes equations.

# 4 The Energetic Implications of the Time Discretisation

This section is concerned with establishing a class of time discretisations which inherit the self–same energetic properties (nonlinear dissipation in the absence of forcing and long–term stability under conditions of time dependent loading) as the analytic problem, irrespective of the time increment employed. In this section a generalised, Euler difference time–stepping scheme for the completely general reference equation is formulated and the energetic implications are investigated in a similar manner to that carried out for the analytic equations in the previous section.

This stability analysis is inspired by the approach of others to schemes for the conventional Navier–Stokes equations. The desirability of the attributes identified as key energetic properties is recognised and they have been used as a benchmark in the analysis of various of the conventional, Eulerian Navier–Stokes schemes by a host of authors. Related work on the conventional, Eulerian Navier–Stokes equations can be found in a variety of references, for example Temam [17] and Simo and Armero [14].

The analyses presented here are extended, not only in the sense that they deal with the completely general reference equation, but also in that non–zero boundaries, so–called free boundaries and time–dependent loads are able to be taken into account (the former two as a consequence of the new inequality). The findings of this work have profound consequences for the implementation of the deforming reference equations. It is significant that many algorithms used for long–term simulation do not automatically inherit the fundamental qualitative features of the dynamics.

### A Generalised Time–Stepping Scheme

An expression for a generalised Euler difference time–stepping scheme can be formulated by introducing an "intermediate" velocity

$$\tilde{\boldsymbol{v}}_{n+\alpha} \equiv \alpha\ \tilde{\boldsymbol{v}}\ |_{t+\Delta t} + (1-\alpha)\ \tilde{\boldsymbol{v}}\ |_{t} \qquad \text{for} \qquad \alpha \in [0,1] \tag{14}$$

to the variational momentum equation (equation (8) on page 7) where $\tilde{\boldsymbol{v}}\ |_t$ and $\tilde{\boldsymbol{v}}\ |_{t+\Delta t}$ are the solutions at times $t$ and $t+\Delta t$ respectively, $\Delta t$ being the time step. It is in this way that a generalised time–discrete approximation of the momentum equation,

$$\begin{aligned}
\frac{\rho}{\Delta t}\left\langle \tilde{\boldsymbol{w}}, (\tilde{\boldsymbol{v}}_{n+1} - \tilde{\boldsymbol{v}}_n)\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} = &\\
&\left\langle \tilde{p}\tilde{\nabla}\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{F}}^{-t}_{n+\alpha}\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} - 2\mu\left\langle \tilde{\boldsymbol{D}}(\tilde{\boldsymbol{w}}), \tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}}_{n+\alpha})\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})}\\
&-\rho\left\langle \tilde{\boldsymbol{w}}, (\tilde{\nabla}\tilde{\boldsymbol{v}}_{n+\alpha})\tilde{\boldsymbol{F}}^{-1}_{n+\alpha}\left(\tilde{\boldsymbol{v}}_{n+\alpha} - \tilde{\boldsymbol{v}}^{ref}_{n+\alpha}\right)\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})}\\
&+\rho\left\langle \tilde{\boldsymbol{w}}, \tilde{\boldsymbol{b}}_{n+\alpha}\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} + \left\langle \tilde{\boldsymbol{w}}, \tilde{\boldsymbol{P}}_{n+\alpha}\tilde{\boldsymbol{N}}_{n+\alpha}\right\rangle_{L^2(\tilde{\Gamma}_{n+\alpha})},
\end{aligned} \tag{15}$$

is derived, where $\langle\ .\ \rangle_{L^2(\tilde{\Omega}_{n+\alpha})}$ denotes the $L^2$ inner product over the deforming domain at time $t+\alpha\Delta t$. $\tilde{\Gamma}_{n+\alpha}$, $\tilde{\boldsymbol{F}}_{n+\alpha}$, $\tilde{J}_{n+\alpha}$, $\tilde{\boldsymbol{D}}_{n+\alpha}$, $\tilde{\boldsymbol{P}}_{n+\alpha}$, and $\tilde{\boldsymbol{b}}_{n+\alpha}$ are likewise defined to be the relevant quantities evaluated at time $t+\alpha\Delta t$.

It will presently become apparent that relevant energy terms are not readily recovered from the time-discrete equations for deforming references in general. It may therefore make sense to perform the analyses for the time–discrete equation in the context of divergence free rates of reference deformation only. A practically less restrictive alternative is too labour intensive. This investigation is accordingly restricted to a subclass of reference deformations in which "reference volume" is conserved. This is for reasons of expedience alone and it is hoped that this subclass of deformations is thought to be representative.

Assumption 1 *The assumptions $\tilde{J}_n = \tilde{J}_{n+\alpha}$ and $\tilde{J}_{n+1} = \tilde{J}_{n+\alpha}$ are made so that the desired energy terms are readily recovered as*

$$\tilde{K}(\tilde{\boldsymbol{v}}_n) = \frac{1}{2}\rho\left|\left|\tilde{\boldsymbol{v}}_n\tilde{J}_n^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega}_n)} = \frac{1}{2}\rho\left|\left|\tilde{\boldsymbol{v}}_n\tilde{J}_{n+\alpha}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})}$$

*and*

$$\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) = \frac{1}{2}\rho\left|\left|\tilde{\boldsymbol{v}}_{n+1}\tilde{J}_{n+1}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega}_{n+1})} = \frac{1}{2}\rho\left|\left|\tilde{\boldsymbol{v}}_{n+1}\tilde{J}_{n+\alpha}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})}.$$

Remark: Notice that $\dfrac{\tilde{J}_{n+1} - \tilde{J}_n}{\Delta t} = \tilde{J}_{n+\alpha}\,\mathrm{div}\,\boldsymbol{v}^{ref}_{n+\alpha}$, the discrete form of $\dfrac{\partial\tilde{J}}{\partial t} = \tilde{J}\,\mathrm{div}\,\boldsymbol{v}^{ref}$, can consequently be rewritten as

$$\mathrm{div}\,\boldsymbol{v}^{ref}_{n+\alpha} = 0$$

under the conditions of the above assumption. It is for the practical expedience afforded by Assumption 1 alone that this analysis is limited to instances in which $\mathrm{div}\,\boldsymbol{v}^{ref}_{n+\alpha} = 0$.

The following lemma will facilitate the elimination of the rate of energy change associated with the convective term under these conditions.

LEMMA 6 (DISCRETE CONVECTIVE ENERGY RATE) *The following relation involving the discrete convective term holds for an incompressible fluid under circumstances of* $\operatorname{div} \boldsymbol{v}^{ref}_{n+\alpha} = 0$*:*

$$-\rho \left\langle \tilde{\boldsymbol{v}}_{n+\alpha}, (\tilde{\nabla} \tilde{\boldsymbol{v}}_{n+\alpha}) \tilde{\boldsymbol{F}}^{-1}_{n+\alpha} \left( \tilde{\boldsymbol{v}}_{n+\alpha} - \tilde{\boldsymbol{v}}^{ref}_{n+\alpha} \right) \tilde{J}_{n+\alpha} \right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})}$$
$$= -\frac{1}{2} \rho \left\langle \tilde{\boldsymbol{v}}_{n+\alpha}, \tilde{\boldsymbol{v}}_{n+\alpha} \left( \tilde{\boldsymbol{F}}^{-t}_{n+\alpha} \tilde{\boldsymbol{N}} \cdot (\tilde{\boldsymbol{v}}_{n+\alpha} - \tilde{\boldsymbol{v}}^{ref}_{n+\alpha}) \right) \tilde{J}_{n+\alpha} \right\rangle_{L^2(\tilde{\Gamma}_{n+\alpha})}.$$

*(Proof in Appendix II.)*

REMARK: Recall that in the investigation of the analytic problem, a term arising from the manipulation of the acceleration containing term (the term containing the rate of change of the Jacobian) cancelled with the convective energy. It is therefore not surprising that assumptions pertaining to the acceleration containing term (in particular to the rate of change of the Jacobian) in the discrete problem will, once made, also be necessary for the corresponding discrete convective energy term to vanish (reffering to the $\operatorname{div} \boldsymbol{v}^{ref} = 0$ condition of Lemma 6). This is a good prognosis for the energetic behaviour of the discrete problem in circumstances of reference deformations excluded by Assumption 1. The full ramifications of Assumption 1 are considered in Subsection 8.2 of Appendix I. This concludes the preliminaries required for the analysis of the time–discrete equation.

## 4.1 Nonlinear Dissipation in the Absence of Forcing

The following analysis establishes a class of time–stepping schemes which exhibit nonlinear dissipation in the absence of forcing regardless of the time increment employed.

THEOREM 3 (NONLINEAR DISSIPATION IN THE ABSENCE OF FORCING) *Suppose that the description is of a vanishing* $\tilde{\boldsymbol{n}}_{n+\alpha} \cdot \left( \tilde{\boldsymbol{v}}_{n+\alpha} - \tilde{\boldsymbol{v}}^{ref}_{n+\alpha} \right)$ *type at free boundaries, purely Eulerian at boundaries across which there is an imposed velocity and that the deformation rate of the reference is divergence free. A sufficient condition for the kinetic energy associated with the generalised class of time–stepping schemes to decay nonlinearly*

$$\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) - \tilde{K}(\tilde{\boldsymbol{v}}_n) \quad \leq \quad -\Delta t \; 2\mu \left\| \tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}}_{n+\alpha}) \tilde{J}^{\frac{1}{2}}_{n+\alpha} \right\|^2_{L^2(\tilde{\Omega}_{n+\alpha})}$$

*in the absence of forcing and irrespective of the time increment employed, is that the scheme is as, or more, implicit than central difference. That is*

$$\alpha \geq \frac{1}{2}.$$

PROOF: Expressing the "intermediate" velocities $\tilde{\boldsymbol{v}}_{n+\frac{1}{2}}$ and $\tilde{\boldsymbol{v}}_{n+\alpha}$ in terms of equation (14) and subtracting, the result

$$\tilde{\boldsymbol{v}}_{n+\alpha} = \left( \alpha - \frac{1}{2} \right) (\tilde{\boldsymbol{v}}_{n+1} - \tilde{\boldsymbol{v}}_n) + \tilde{\boldsymbol{v}}_{n+\frac{1}{2}} \tag{16}$$

is obtained. The first step towards formulating an expression involving the kinetic energy of the generalised time stepping–scheme (15) is to replace the arbitrary vector, $\boldsymbol{w}$, with $\tilde{\boldsymbol{v}}_{n+\alpha}$. By further substituting (16) into (15) and eliminating the pressure containing term on the basis of incompressibility (equation (7)), an expression involving the difference in kinetic energy over the duration of a single time step is obtained.

Incompressibility and a restriction on reference deformations to those for which $\mathrm{div}\boldsymbol{v}^{ref}_{n+\alpha}$ is zero ensure that the Lemma 6 condition is satisfied.

The equation

$$\begin{aligned}\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) - \tilde{K}(\tilde{\boldsymbol{v}}_n) \;=\; & -\rho\left(\alpha-\frac{1}{2}\right)\left\|\left(\tilde{\boldsymbol{v}}_{n+1}-\tilde{\boldsymbol{v}}_n\right)\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right\|^2_{L^2(\tilde{\Omega}_{n+\alpha})} \\ & -\Delta t\; 2\mu\left\|\tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}}_{n+\alpha})\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right\|^2_{L^2(\tilde{\Omega}_{n+\alpha})} + \Delta t\rho\left\langle\tilde{\boldsymbol{v}}_{n+\alpha},\tilde{\boldsymbol{b}}_{n+\alpha}\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} \\ & +\Delta t\left\langle\tilde{\boldsymbol{v}}_{n+\alpha},\tilde{\boldsymbol{P}}_{n+\alpha}\tilde{\boldsymbol{N}}_{n+\alpha}\right\rangle_{L^2(\tilde{\Gamma}_{n+\alpha})} \\ & -\Delta t\frac{1}{2}\rho\left\langle\tilde{\boldsymbol{v}}_{n+\alpha},\tilde{\boldsymbol{v}}_{n+\alpha}\left(\tilde{\boldsymbol{F}}^{-t}_{n+\alpha}\tilde{\boldsymbol{N}}\cdot\left(\tilde{\boldsymbol{v}}_{n+\alpha}-\tilde{\boldsymbol{v}}^{ref}_{n+\alpha}\right)\right)\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Gamma}_{n+\alpha})} \end{aligned} \quad (17)$$

is then obtained. The term $\tilde{\boldsymbol{F}}^{-t}_{n+\alpha}\tilde{\boldsymbol{N}}_{n+\alpha}\cdot(\tilde{\boldsymbol{v}}_{n+\alpha}-\tilde{\boldsymbol{v}}^{ref}_{n+\alpha})$ vanishes at fixed impermeable boundaries since both $\tilde{\boldsymbol{F}}^{-t}_{n+\alpha}\tilde{\boldsymbol{N}}_{n+\alpha}\cdot\tilde{\boldsymbol{v}}_{n+\alpha}$ and $\tilde{\boldsymbol{v}}^{ref}_{n+\alpha}$ vanish under such circumstances (assuming the description becomes purely Eulerian there). This self–same term also vanishes at free boundaries according to Lemma 3. Boundaries of an imposed velocity type need not be accounted for as a consequence of the stated "no forcing" condition, and so

$$\begin{aligned}\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) - \tilde{K}(\tilde{\boldsymbol{v}}_n) \;\leq\; & -\rho\left(\alpha-\frac{1}{2}\right)\left\|\left(\tilde{\boldsymbol{v}}_{n+1}-\tilde{\boldsymbol{v}}_n\right)\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right\|^2_{L^2(\tilde{\Omega}_{n+\alpha})} \\ & -\Delta t\; 2\mu\left\|\tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}}_{n+\alpha})\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right\|^2_{L^2(\tilde{\Omega}_{n+\alpha})},\end{aligned}$$

because of this condition. Thus the kinetic energy inherent to the algorithmic flow decreases nonlinearly in the absence of forcing, irrespective of the time increment employed and for arbitrary initial conditions provided that

$$\alpha \geq \frac{1}{2} \qquad \text{and} \qquad \mathrm{div}\,\boldsymbol{v}^{ref}_{n+\alpha} = 0.$$

The former requirement translates directly into one specifying the use of schemes as, or more, implicit than central difference. Only for descriptions which are divergence free has it here been guaranteed that energy will not be artificially introduced by way of the reference.

Remark: Notice (by Lemma 1) that for $\alpha = \frac{1}{2}$ an identical rate of energy decay

$$\frac{\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) - \tilde{K}(\tilde{\boldsymbol{v}}_n)}{\Delta t} \;\leq\; -2\nu C\tilde{K}(\tilde{\boldsymbol{v}}_{n+\alpha})$$

is obtained for the discrete approximation as was obtained for the equations.

## 4.2 Long–Term Stability under Conditions of Time–Dependent Loading

This second part of the time–discrete analysis establishes a class of time stepping schemes which exhibit long–term stability under conditions of time dependent loading, irrespective of the time increment employed. The following lemma is necessary to the analysis and is concerned with devising a bound for the energy at an intermediate point in terms of energy values at either end of the time step.

LEMMA 7 (INTERMEDIATE POINT ENERGY) *The following bound applies*

$$\tilde{K}(\tilde{\boldsymbol{v}}_{n+\alpha}) \geq \alpha\,(\alpha - c + \alpha c)\,\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) + (1-\alpha)\left(1 - \alpha - \frac{\alpha}{c}\right)\tilde{K}(\tilde{\boldsymbol{v}}_{n})$$

*where $c$ is some constant, $c > 0$.*

The optimal choice of the constant $c$ is established farther on. The following theorem establishes a class of time–stepping schemes which exhibit long–term stability under conditions of time–dependent loading regardless of the time increment employed.

THEOREM 4 (LONG–TERM STABILITY) *Suppose that the description is of a vanishing $\tilde{\boldsymbol{n}}_{n+\alpha} \cdot \left(\tilde{\boldsymbol{v}}_{n+\alpha} - \tilde{\boldsymbol{v}}^{ref}_{n+\alpha}\right)$ type at free boundaries, that it becomes purely Eulerian at fixed impermeable boundaries and that the rate at which the reference is deformed is divergence free. A sufficient condition for the algorithmic flow to exhibit long–term stability under conditions of time–dependent loading assuming this time–dependent loading and the speed of the free surface is bounded in such a way that*

$$\rho\left|\left|\tilde{\boldsymbol{b}}_{n+\alpha}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})} + \left|\left|\tilde{\boldsymbol{P}}_{n+\alpha}\tilde{\boldsymbol{N}}_{n+\alpha}\right|\right|^2_{L^2(\tilde{\Gamma}_{n+\alpha})} + \nu^2 C^2\left|\left|\tilde{\boldsymbol{v}}_{n+\alpha}\right|\right|^2_{L^2(\tilde{\Gamma}_{n+\alpha})} \quad \leq \quad M^2, \qquad ^\dagger$$

*is*

$$\alpha > \frac{1}{2}.$$

PROOF: Substituting Lemmas 1, 3 and 4 into equation (17), applying the above bound and choosing $\alpha \geq \frac{1}{2}$ one obtains

$$\frac{\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) - \tilde{K}(\tilde{\boldsymbol{v}}_{n})}{\Delta t} + \nu C\tilde{K}(\tilde{\boldsymbol{v}}_{n+\alpha}) \quad \leq \quad \frac{M^2}{2\nu C}.$$

From this point on the argument used is identical to that of SIMO and ARMERO [14] for the conventional, Eulerian Navier–Stokes equations. Substitution of Lemma 7 leads to a recurrence relation,

$$\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) \quad \leq \quad \frac{1 - \nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\Delta t}{1 + \nu C\alpha(\alpha - c + \alpha c)\Delta t}\tilde{K}(\tilde{\boldsymbol{v}}_{n}) + \frac{M^2\Delta t}{2\nu C\left[1 + \nu C\alpha(\alpha - 1 + \alpha c)\Delta t\right]}.$$

$^\dagger$Note that this bound does not incorporate a contribution from boundaries of an imposed velocity type in any obvious way. The two boundary terms are only applicable at boundaries which involve tractions.

Using this recurrence relation to take cognisance of the energy over all time steps,

$$
\begin{aligned}
\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) \leq & \left[\frac{1-\nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\Delta t}{1+\nu C\alpha(\alpha-c+\alpha c)\Delta t}\right]^n \tilde{K}(\tilde{\boldsymbol{v}}_0) \\
& + \frac{M^2 \Delta t}{2\nu C\left[1+\nu C\alpha(\alpha-c+\alpha c)\Delta t\right]} \sum_{k=0}^{n-1} \left[\frac{(1-\nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\Delta t)}{1+\nu C\alpha(\alpha-c+\alpha c)\Delta t}\right]^k
\end{aligned} \tag{18}
$$

is obtained. An infinite geometric series which converges so that

$$
\begin{aligned}
\lim_{n\to\infty} \sup \tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) \leq & \frac{M^2 \Delta t}{2\nu C\left[1+\nu C\alpha(\alpha-c+\alpha c)\Delta t\right]} \Bigg[1 \\
& - \frac{(1-\nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\Delta t)}{1+\nu C\alpha(\alpha-c+\alpha c)\Delta t}\Bigg]^{-1} \\
= & \frac{M^2}{2\nu C\left[\nu C\alpha(\alpha-c+\alpha c)+\nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\right]}
\end{aligned}
$$

results, providing the absolute ratio of the series is less than unity. That is

$$
\left|\frac{1-\nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\Delta t}{1+\nu C\alpha(\alpha-c+\alpha c)\Delta t}\right| < 1.
$$

Therefore either

$$
-1-\nu C\alpha(\alpha-c+\alpha c)\Delta t < 1-\nu C(1-\alpha)\left(1-\alpha-\frac{\alpha}{c}\right)\Delta t
$$

or

$$
1-\nu C(1-\alpha)\left(1-\alpha-\frac{\alpha}{c}\right)\Delta t < 1+\nu C\alpha(\alpha-c+\alpha c)\Delta t \tag{19}
$$

in order for the bound to exist. Notice, furthermore, that for this desired convergence to be unconditional (regardless of the time increment employed) requires

$$
\alpha - c + \alpha c \geq 0. \tag{20}
$$

The denominator in the series ratio might otherwise vanish for some value of $\Delta t$.

For $\alpha \in \left[\frac{1}{2}, 1\right]$ equation (19) and equation (20) together imply

$$
\frac{(1-\alpha)}{\alpha} < c \leq \frac{\alpha}{(1-\alpha)}
$$

which in its turn implies

$$
\frac{(1-\alpha)}{\alpha} < \frac{\alpha}{(1-\alpha)}.
$$

The choice of the parameter $\alpha > \frac{1}{2}$ therefore leads to an infinite geometric series which forms the desired upper bound. The minimum value of this bound occurs for $c$ chosen according to

$$\inf_{\frac{(1-\alpha)}{\alpha}<c\leq\frac{\alpha}{(1-\alpha)}} \frac{1}{\nu C\alpha(\alpha - c + \alpha c) + \nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})} = \frac{1}{\nu C(2\alpha-1)^2}.$$

The value of this upper bound, which occurs for the choice of the parameter $\alpha > \frac{1}{2}$, is then

$$\lim_{n\to\infty} \sup \tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) \leq \frac{M^2}{2\nu^2 C^2 (2\alpha-1)^2}.$$

In this way one arrives at a class of algorithms which are unconditionally (irrespective of the time increment employed) stable.

Remark: Notice that for $\alpha = 1$ one obtains an identical energy bound for the discrete approximation as was obtained for the equations.

# 5 Some Numerical Examples

Some numerical results for problems of the type in question are presently given. The theory thus far developed was employed in the simulation of a driven cavity flow, a driven cavity flow with various, included rigid bodies, a die–swell problem and a Stokes, second order wave.

The approach taken when approximating free surfaces, was that they may be treated as a material entity, that is, the material derivative of the free surface was assumed zero. Euler's equations and conservation of linear momentum were used to determine the motion of the rigid body. A predictor–corrector method was used to solve the combined sub–problems.

A backward difference scheme was used to approximate the time derivative in the fluid sub–problem (in compliance with the Theorem 3 and Theorem 4 conditions), the finite element method was used for the spatial (referential "space") discretisation and a $Q_2$–$P_1$ element pair was used as a basis. A penalty method was employed to eliminate pressure as a variable and nonlinearity was circumvented by way of a new, second order accurate linearisation. Linearising with a guess obtained by extrapolating through solutions from the previous two time steps leads to second order accuracy.

Theorem 5 *The linearised terms,* $(2\boldsymbol{v}\mid_t - \boldsymbol{v}\mid_{t-\Delta t})\cdot\nabla\boldsymbol{v}\mid_{t+\Delta t}$ *and* $\boldsymbol{v}\mid_{t+\Delta t}\cdot\nabla(2\boldsymbol{v}\mid_t - \boldsymbol{v}\mid_{t-\Delta t})$*, are second order accurate (have error* $O(\Delta t^2)$*) approximations of the nonlinear term* $(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\mid_{t+\Delta t}$*.*

Proof:

$$\boldsymbol{v}\mid_{t+\Delta t} = \boldsymbol{v}\mid_t + \Delta t\left.\frac{\partial\boldsymbol{v}}{\partial t}\right|_t + O(\Delta t^2) \qquad \textit{(by Taylor series)}$$

$$
\begin{aligned}
&= \boldsymbol{v}\,|_t + \Delta t \left( \frac{\boldsymbol{v}\,|_t - \boldsymbol{v}\,|_{t-\Delta t}}{\Delta t} + O(\Delta t) \right) + O(\Delta t^2) \qquad \textit{(using a backward difference)} \\
&= 2\boldsymbol{v}\,|_t - \boldsymbol{v}\,|_{t-\Delta t} + O(\Delta t^2) \\
(\boldsymbol{v} \cdot \nabla \boldsymbol{v})\,|_{t+\Delta t} &= \left[ 2\boldsymbol{v}\,|_t - \boldsymbol{v}\,|_{t-\Delta t} + O(\Delta t^2) \right] \cdot \nabla \boldsymbol{v}\,|_{t+\Delta t} \\
&= \left[ 2\boldsymbol{v}\,|_t - \boldsymbol{v}\,|_{t-\Delta t} \right] \cdot \nabla \boldsymbol{v}\,|_{t+\Delta t} + O(\Delta t^2)
\end{aligned}
$$

The above linearisation schemes are an improvement on the conventional $\boldsymbol{v}\,|_t \cdot \nabla \boldsymbol{v}\,|_{t+\Delta t}$ † or $\boldsymbol{v}\,|_{t+\Delta t} \cdot \nabla \boldsymbol{v}\,|_t$ linearisation schemes by an order of magnitude. A detailed exposition of all numerical methods otherwise used in these simulations can be found in Childs and Reddy [1].

## 5.1 Example 1: Driven Cavity Flow

The problem is essentially that of a square, two–dimensional pot whose lid is moved across the top at a rate equal to its diameter for a Reynolds number of unity. The boundary conditions are accordingly "no slip" on container walls and a horizontal flow of unity across the top (depicted in Fig. 3).

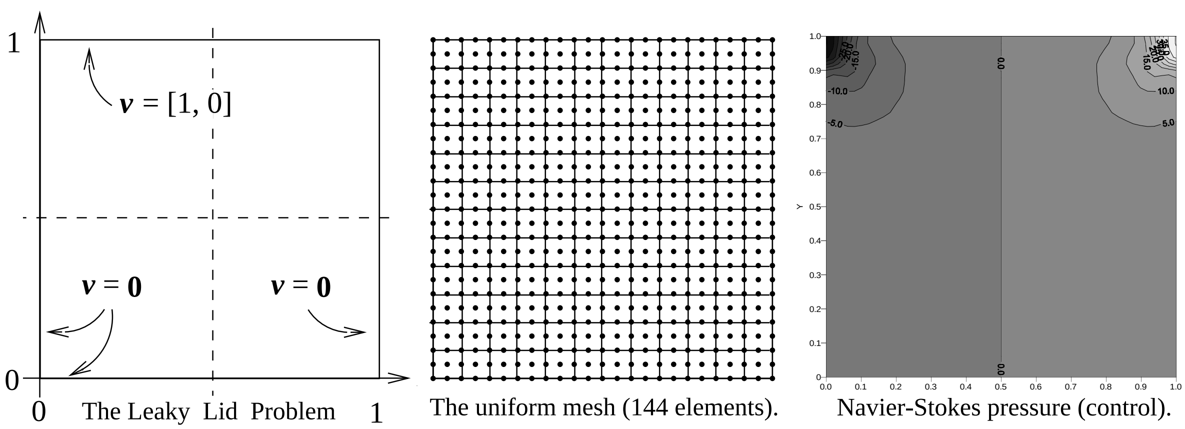
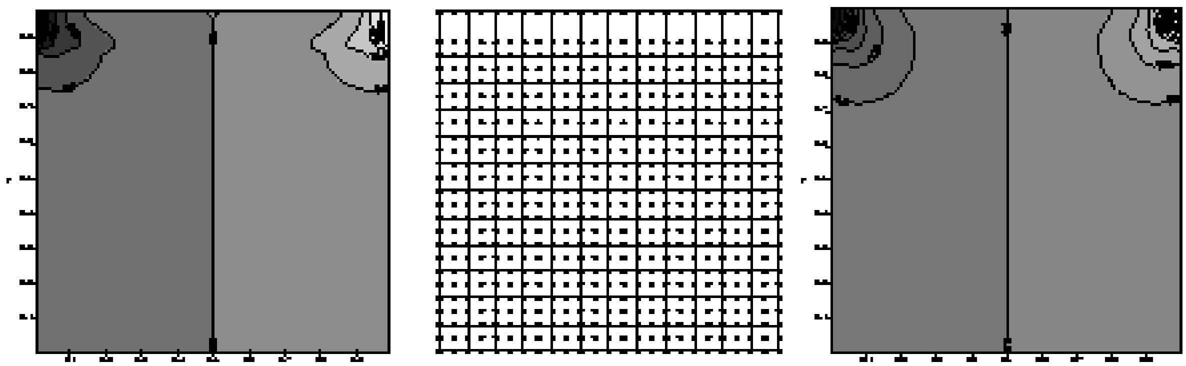



Figure 3: The Problem, the Mesh and the Pressures Obtained Using the Conventional Eulerian Equations.

The idea here was to compare results obtained using the completely general reference equation on a deforming mesh with those obtained using the conventional, Eulerian Navier–Stokes equations.

†Favoured in terms of both rate and radius of convergence by Cuvelier, Segal and van Steenhoven [3].

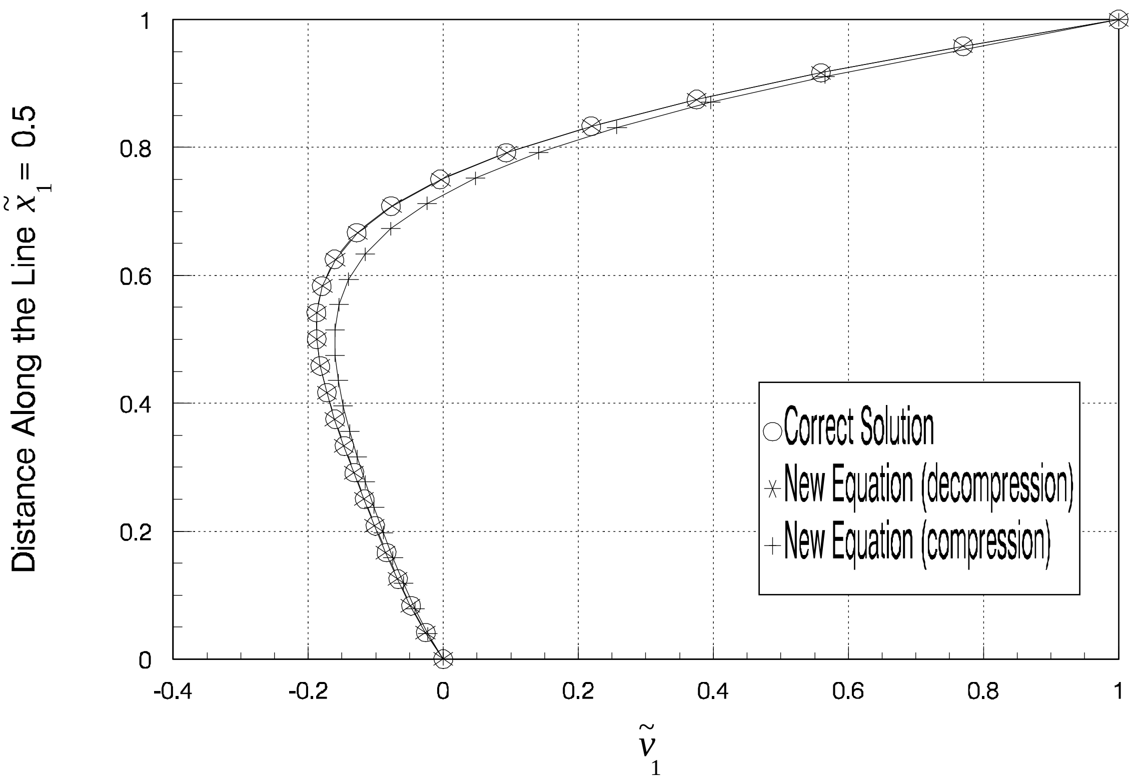


Pressure for a 5% mesh compress-ession. | The distorted mesh (144 elements) "compressed" by 5%. | Pressure for a 5% mesh decompre-ssion.

Figure 4: Pressures Obtained Using the Completely General Reference Equation and a Deforming Mesh.

The corresponding velocity profiles along the cuts depicted in Fig. 3 are given in Fig. 5.

Velocity Profiles



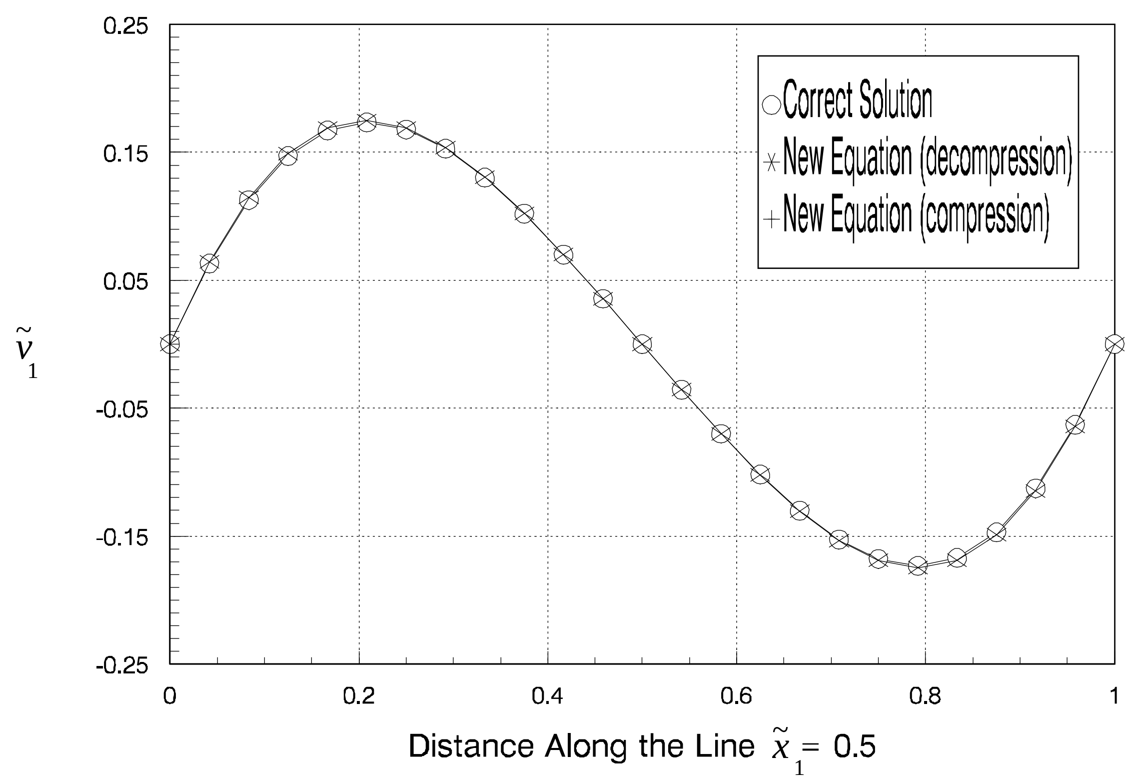




Figure 5: In this test part of the mesh was successively compressed and decompressed by 5 % over two time steps of length 0.05.

## 5.2 Example 2: "Pebble in a Pothole"

In this example rigid bodies of varying mass and moments of inertia were released from rest in a flow dictated by the same boundary conditions as the driven cavity flow of the previous example. One would expect a die bead (a small rigid body of neutral bouyancy) to move in tandem with the fluid soon after its release from rest. One might also expect a clockwise rotation to be induced by concentrating the mass closer to the centre i.e. lowering the moment of inertia.

The finite element mesh was automatically generated and adjusted about the included rigid body in what is possibly a slightly novel fashion. A small region of mesh immediately adjacent to the included rigid body was repeatedly remapped to cope with the changing orientation, the remainder was squashed/stretched according to the translation.

To begin with, a square region of mesh centered on, and including the rigid body, is deleted (depicted in Fig. 6). Each of four wedge–shaped regions is then demarcated (the intersections of lines which bisect corners and edges of the square frame, with the surface of the rigid body are located using Newton's method) by as many points as there are nodes in an element i.e. each wedge shaped–region is set up as a massive element.

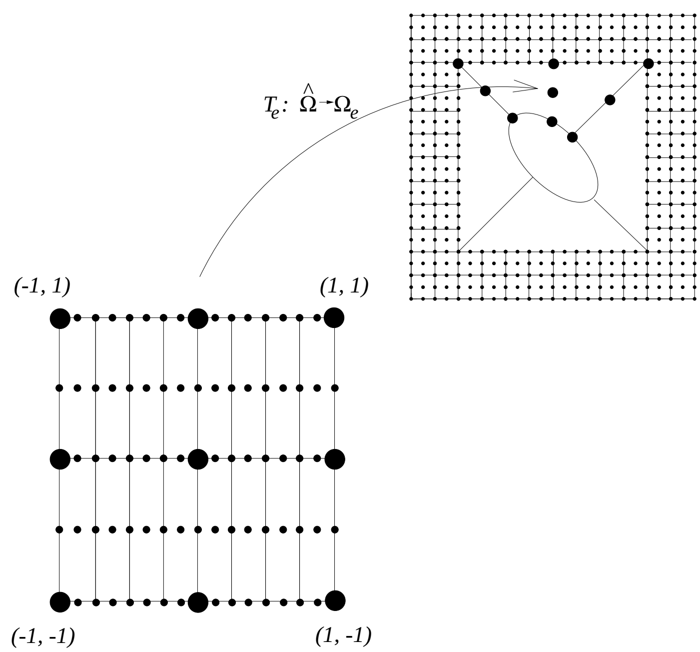



Figure 6: The Local Distortion is Obtained by Mapping Square Chunks of Rectangular Mesh Using Finite Element Mappings.

Chunks of uniform mesh, which have identical extremities to those of the master element, are then mapped into the newly–demarcated, wedge–shaped regions using finite element mappings (in exactly the same manner as points in the master element domain are,

in theory, mapped into individual mesh elements). Further, fine adjustment of nodes intended to delineate the surface of the rigid body is accomplished by moving them along a line between node and centre, to the rigid body surface using Newton's method. The mesh outside the "box" (the box containing the 4 wedges enclosing the rigid body) is squashed/stretched according to the requirements of the translation (the nodes are translated by a factor inversely proportional to their distance from the box). This method satisfies the requirement that $\tilde{\boldsymbol{n}} \cdot \left(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}\right)$ vanishes at the fluid–rigid body interface (a condition in Theorems 1, 2, 3 and 4). Mesh refinement in the vicinity of the included, rigid body is an automatic by-product of this method.

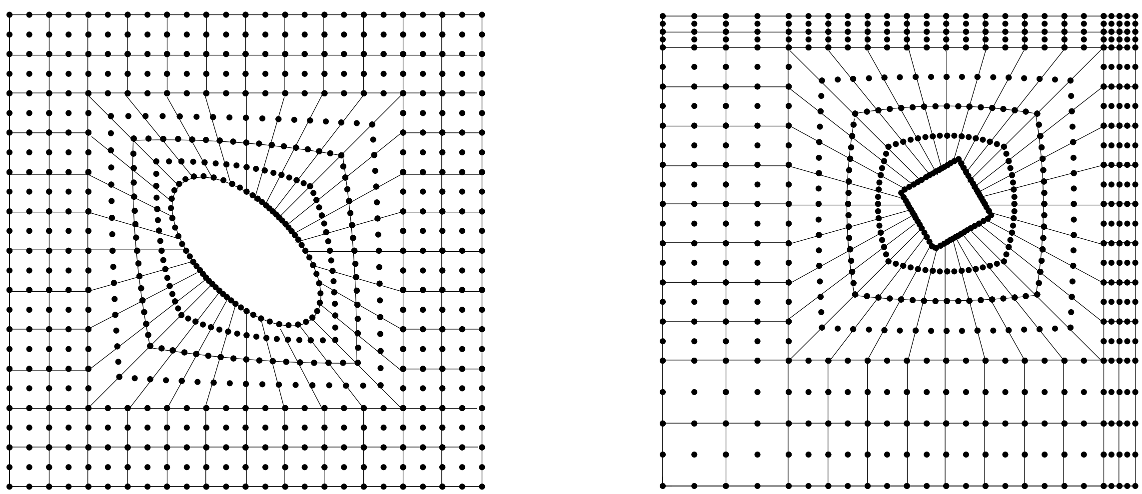


Figure 7: Typical meshes which result when using this method of automatic mesh generation about rigid bodies which are simultaneously rotating and translating.

Various rigid bodies were introduced to the driven cavity flow problem described in Subsection 5.1, in the absence of a body force. The results in Fig. 8 involve the ellipse

$$\frac{x_1^2}{2^2} + \frac{x_2^2}{1^2} = 0.025^2,$$

whose major axis is 0.1. The quantities $\bar{m}$ and $\bar{J}_{ii(\text{no sum})}$ are a dimensionless mass and $i$th principle moment of inertia respectively.

Figure 8: The trajectories of various included rigid bodies released from rest at the centre of the driven cavity flow described. Top Left: $Re = 0.025$, $\bar{m} = 251.3$, $\bar{J}_{33} = 314.2$ and $t = 3.6$ secs. Top Right: $Re = 0.025$, $\bar{m} = 251.3$, $\bar{J}_{33} = 1.0$ and $t = 4.0$ secs. Bottom Left: $Re = 0.025$, $\bar{m} = 251.3$, $\bar{J}_{33} = 0.1$ and $t = 3.6$ secs. Bottom Right: $Re = 1$, $\bar{m} = 1$, moment of inertia (scaled) $= 0.1$ and $t = 2.0$ secs.

## 5.3 Example 3: Die Swell Problems

The axis–symmetric die swell (or fluid jet) problem is a free surface problem well documented in the literature (Kruyt, Cuvelier, Segal and Van Der Zanden [9], Omodei [12] and Engelman and Dupret quoted in Kruyt et al.). The basic theme to this problem is the extrusion of a fluid with initial parabolic flow profile from the end of a short nozzle.

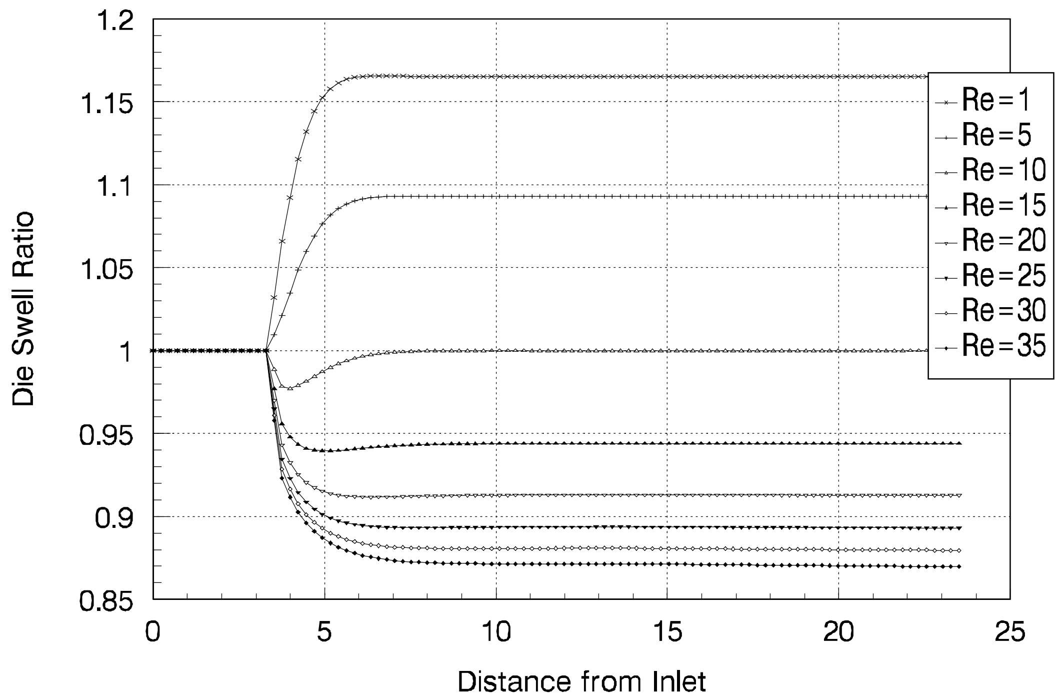



Figure 9: Die swell ratios predicted for various Reynolds numbers using an inlet velocity profile of $\bar{v}_1 = \frac{Re}{1}\frac{3}{2}(1 - \bar{x}_2^2)$ and the methods described. (Bars on the variables merely indicate that they are dimensionless.)

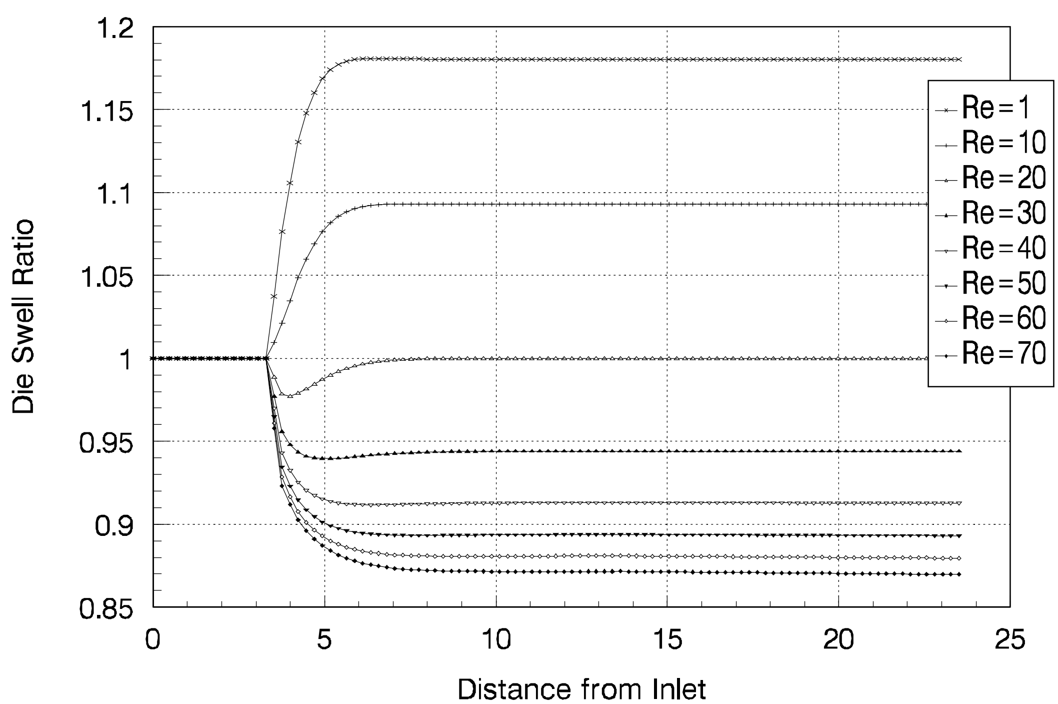



Figure 10: Die swell ratios predicted for various Reynolds numbers using an inlet velocity profile of $\bar{v}_1 = \frac{Re}{10}\frac{3}{2}(1 - \bar{x}_2^2)$ and the methods described.

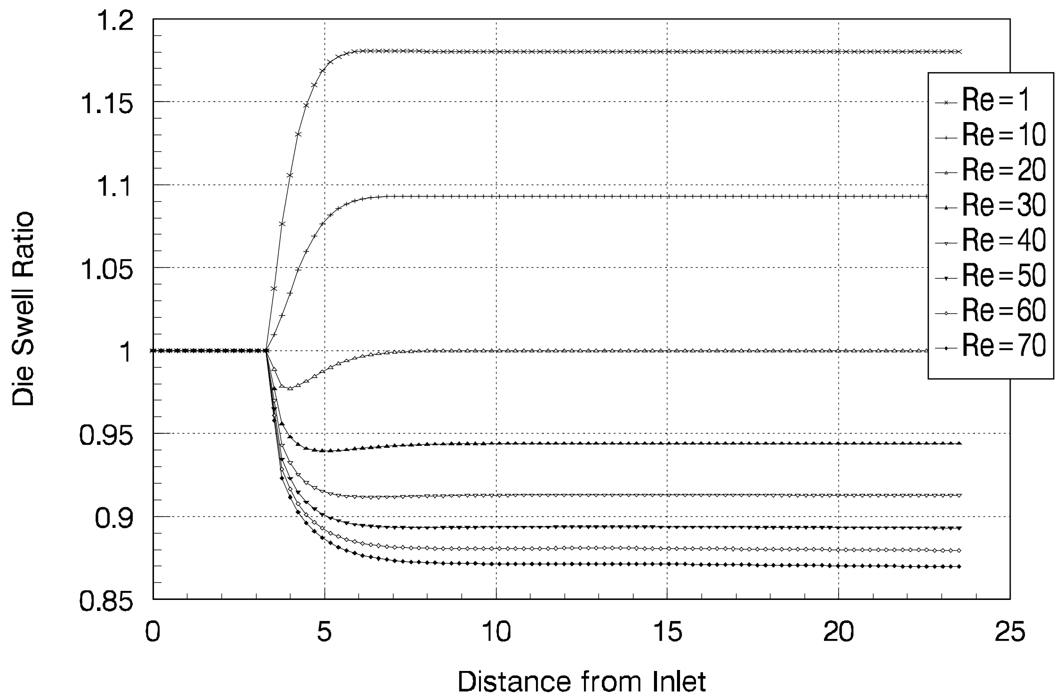



Figure 11: Die swell ratios predicted for various Reynolds numbers using an inlet velocity profile of $\bar{v}_1 = \frac{Re}{20}\frac{3}{2}(1 - \bar{x}_2^2)$ and the methods described.

## 5.4 Example 4: A Stokes Second Order Wave

In this problem the velocity profile and surface elevation predicted by Stokes second order wave theory (see Koutitas [8]) were used as boundary conditions for flow and free surface subproblems respectively.

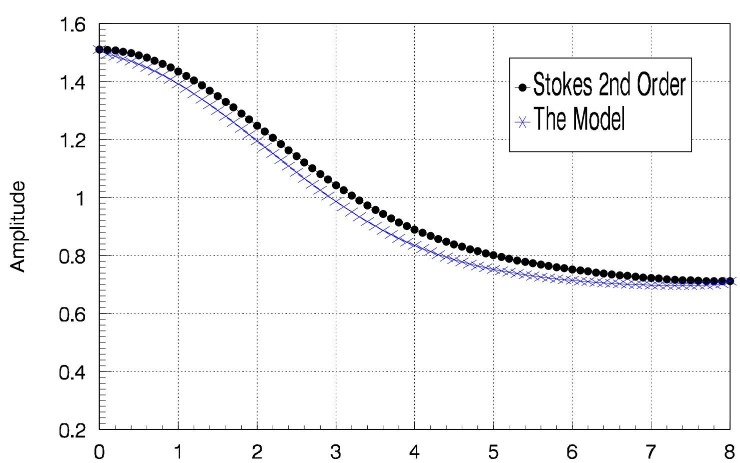



Figure 12: A Stokes second order Wave.

Problems with wave propagation were subsequently experienced as time progressed. It should be noted, however, that the problem was not attempted with the same seriousness as previous examples and the mesh was poor (there being only three elements in the vertical extent of the mesh).

# 6 Conclusions

The correct equations, which describe the motion of an incompressible, Newtonian fluid and which are valid for a completely general range of reference deformations, are equations (6) and (7). For implementations requiring the equations to be evaluated about a single instant within each time step only (eg. finite differences), the deformation gradients may be assumed identity i.e. the equations of Hughes, Liu and Zimmerman [6] (equations (10) and (11)) will suffice.

In this work it is shown (as was hoped) that nonlinear, exponential–type dissipation in the absence of forcing and long–term stability under conditions of time dependent loading are properties automatically inherited by such deforming reference descriptions. The single provisor is that the conventional boundary descriptions are used (vanishing $\tilde{\boldsymbol{n}} \cdot \left(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}\right)$ type at free boundaries, purely Eulerian at boundaries across which there is an imposed velocity or where boundaries are of a fixed, impermeable type). These properties are intrinsic to real flows and the conventional, Eulerian Navier–Stokes equations.

Relevant energy terms are, however, not readily recovered from the time–discrete equations for deforming references in general. Only for divergence free rates of reference deformation could it consequently be guaranteed that energy would not be artificially introduced to the algorithmic flow by way of the reference. A further casualty of the discrete analysis is its failure to account for flows driven by their boundaries in any obvious way i.e. boundaries of an imposed velocity type do not enter explicitly into the bound. Scope for the further development of this work therefore exists.

The divergence free assumption was made for reasons of expedience alone and it is hoped that the findings of the time–discrete analysis can be extrapolated to a more general class of mesh deformations. If one were to be overly cautious on this basis one would be faced with the additional challenge of enforcing mesh deformations which are divergence free. Such a totally divergence free description may, however, not be practical. Both the purely Lagrangian and purely Eulerian fluid descriptions have divergence free rates of distortion.

What is clear is that there are inherent problems with using certain classes of time–stepping schemes and the use of finite difference schemes more implicit than central difference is consequently advocated. The limitations of the time–discrete analysis do not detract from this finding in any way. Such differences exhibit the key energetic properties (nonlinear, exponential–type dissipation in the absence of forcing and long–term stability under conditions of time dependent loading) irrespective of the time increment employed. A backward difference is the obvious choice. Calculations at time $t+\alpha\Delta t$ would require an intermediate mesh and associated quantities for instances in which $\alpha \neq 1$ (since $\alpha > \frac{1}{2}$).

The author recommends a strategy in which a predominantly Eulerian description is used, where possible, for the bulk of the problem (from an efficiency point of view) and the completely general reference description for the remainder is appropriate. Purely Eulerian descriptions have the advantage of a "one off" finite element construction and involve none of the hazards of a badly distorted reference.

With regard to numerical implementation and using a $Q_2$–$P_1$ element pair, it was found that pressures approximated as linear on the master element still led to so–called "locking" or "chequerboard" modes. The pressures needed to be linear on the actual elements themselves. This finding makes sense if one considers that a linear function mapped from the master element using a $Q_2$ mapping will no longer be $P_1$ for non–rectangular elements (the $Q_2$–$P_1$ element pair was shown to satisfy the L.B.B. condition in the context of rectangular elements).

Lastly, the linearised terms $(2\boldsymbol{v}\mid_t - \boldsymbol{v}\mid_{t-\Delta t}) \cdot \nabla \boldsymbol{v}\mid_{t+\Delta t}$ and $\boldsymbol{v}\mid_{t+\Delta t} \cdot \nabla(2\boldsymbol{v}\mid_t - \boldsymbol{v}\mid_{t-\Delta t})$ are second order accurate approximations of the convective term and a remarkably practical, simple and effective method to automatically generate meshes about included rigid bodies was devised.

# 7 Acknowledgements

Daya Reddy is thanked for the loan of references and related advice. Grzegorz Lubczonok and Ronald Becker are thanked for their respective opinions on the inequality (Inequality 1). Others who assisted the author in various ways (including the provision of resources and moral support) are Kevin Colville, George Ellis and Rait Harnett. The final arrangement and editing of this work was carried out under the supervision of Philip Gresho.

# 8 Appendix I

## 8.1 The Contribution $\tilde{K}_{\tilde{\Gamma}_{\text{imposed } \tilde{v}}}$ at Imposed Velocity–Type Boundaries

Additional terms which arise from the limits of the integral

$$\rho \int_{\tilde{x}_1^0}^{\tilde{x}_1'} \int_{\tilde{x}_2^0}^{\tilde{x}_2'} \int_{\tilde{x}_3^0}^{\tilde{x}_3'} \tilde{\boldsymbol{v}} \cdot \frac{\partial \tilde{\boldsymbol{v}}}{\partial t} \tilde{J} \; d\tilde{x}_3 d\tilde{x}_2 d\tilde{x}_1$$

when changing the order of differentiation and integration at boundaries across which there is an imposed velocity are:

$$-\frac{1}{2}\rho \left[ \frac{D\tilde{x}_1'}{Dt} \left( \int_{\tilde{x}_2^0}^{\tilde{x}_2'} \int_{\tilde{x}_3^0}^{\tilde{x}_3'} \tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{v}} \tilde{J} d\tilde{x}_3 d\tilde{x}_2 \right) \Bigg|_{\tilde{x}_1'} - \frac{D\tilde{x}_1^0}{Dt} \left( \int_{\tilde{x}_2^0}^{\tilde{x}_2'} \int_{\tilde{x}_3^0}^{\tilde{x}_3'} \tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{v}} \tilde{J} d\tilde{x}_3 d\tilde{x}_2 \right) \Bigg|_{\tilde{x}_1^0}\right.$$

$$+\int_{\tilde{x}_1^0}^{\tilde{x}_1'} \frac{D\tilde{x}_2'}{Dt}\left(\int_{\tilde{x}_3^0}^{\tilde{x}_3'} \tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{v}}\tilde{J}d\tilde{x}_3\right)\Big|_{\tilde{x}_2'} d\tilde{x}_1 - \int_{\tilde{x}_1^0}^{\tilde{x}_1'} \frac{D\tilde{x}_2^0}{Dt}\left(\int_{\tilde{x}_3^0}^{\tilde{x}_3'} \tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{v}}\tilde{J}d\tilde{x}_3\right)\Big|_{\tilde{x}_2^0} d\tilde{x}_1$$
$$+\int_{\tilde{x}_1^0}^{\tilde{x}_1'}\int_{\tilde{x}_2^0}^{\tilde{x}_2'} \frac{D\tilde{x}_3'}{Dt}\left(\tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{v}}\tilde{J}\right)\Big|_{\tilde{x}_3'} d\tilde{x}_2 d\tilde{x}_1 - \int_{\tilde{x}_1^0}^{\tilde{x}_1'}\int_{\tilde{x}_2^0}^{\tilde{x}_2'} \frac{D\tilde{x}_3^0}{Dt}\left(\tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{v}}\tilde{J}\right)\Big|_{\tilde{x}_3^0} d\tilde{x}_2 d\tilde{x}_1\Bigg] \tag{21}$$

(using Leibnitz's rule for differentiation under the integral sign).

Note that the terms $\dfrac{D\dots}{Dt}$ can be given a

$$\frac{\partial\dots}{\partial t} + \tilde{\nabla}\dots\tilde{\boldsymbol{F}}^{-1}(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref})$$

interpretation in terms of the relation (1) established at the beginning of Section 2.2. For descriptions which are purely Eulerian at such fixed boundaries, $\tilde{\boldsymbol{v}}^{ref}$ vanishes and $\tilde{\boldsymbol{F}}$ is identity. Thus the extra terms, (21) above, can be rewritten (with minus sign omitted)

$$\frac{1}{2}\rho\left[\nabla x_1'\cdot\boldsymbol{v}\left(\int_{x_2^0}^{x_2'}\int_{x_3^0}^{x_3'}\boldsymbol{v}\cdot\boldsymbol{v}\;dx_3dx_2\right)\Big|_{\tilde{x}_1'} - \nabla x_1^0\cdot\boldsymbol{v}\left(\int_{x_2^0}^{x_2'}\int_{x_3^0}^{x_3'}\boldsymbol{v}\cdot\boldsymbol{v}\;dx_3dx_2\right)\Big|_{\tilde{x}_1^0}\right.$$
$$+\int_{x_1^0}^{x_1'}\nabla x_2'\cdot\boldsymbol{v}\left(\int_{x_3^0}^{x_3'}\boldsymbol{v}\cdot\boldsymbol{v}\;dx_3\right)\Big|_{\tilde{x}_2'} dx_1 - \int_{x_1^0}^{x_1'}\nabla x_2^0\cdot\boldsymbol{v}\left(\int_{x_3^0}^{x_3'}\boldsymbol{v}\cdot\boldsymbol{v}\;dx_3\right)\Big|_{\tilde{x}_2^0} dx_1$$
$$\left.+\int_{x_1^0}^{x_1'}\int_{x_2^0}^{x_2'}\nabla x_3'\cdot\boldsymbol{v}\,(\boldsymbol{v}\cdot\boldsymbol{v})|_{\tilde{x}_3'}\;dx_2dx_1 - \int_{x_1^0}^{x_1'}\int_{x_2^0}^{x_2'}\nabla x_3^0\cdot\boldsymbol{v}\,(\boldsymbol{v}\cdot\boldsymbol{v})|_{\tilde{x}_3^0}\;dx_2dx_1\right]$$

i.e. a total rate of energy transport across the boundaries, similar, but not identical to $\int_\Gamma(\boldsymbol{n}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{v})d\Gamma$. These terms, dubbed $\tilde{K}_{\tilde{\Gamma}_{\text{imposed }\tilde{\boldsymbol{v}}}}$ in the work, must be added to the right hand side of equation (13) when circumstances require.

This approach may seem comparatively crude in the light of rather elegant work done by TEMAM [17] for flows driven by their boundaries, however, the intended purpose differs slightly. What is here being sought is a bound formulated in terms of known, physically comprehensible quantities at the boundary which are independant of the solution.

## 8.2 The Ramifications of Assumption 1

If one does not make Assumption 1, contributions from the

$$\frac{\rho}{\Delta t}\left\langle\tilde{\boldsymbol{v}}_{n+\alpha},(\tilde{\boldsymbol{v}}_{n+1}-\tilde{\boldsymbol{v}}_n)\tilde{J}_{n+\alpha}\right\rangle$$

term in equation (15) and

$$\frac{1}{2}\rho\left\langle\tilde{\boldsymbol{v}}_{n+\alpha},\tilde{\boldsymbol{v}}_{n+\alpha}\frac{(\tilde{J}_{n+1}-\tilde{J}_n)}{\Delta t}\right\rangle$$

in Lemma 2 amount to

$$\frac{\rho}{\Delta t}\left\langle\tilde{\boldsymbol{v}}_{n+\alpha},(\tilde{\boldsymbol{v}}_{n+1}-\tilde{\boldsymbol{v}}_n)\tilde{J}_{n+\alpha}\right\rangle + \frac{\rho}{\Delta t}\frac{1}{2}\left\langle\tilde{\boldsymbol{v}}_{n+\alpha},\tilde{\boldsymbol{v}}_{n+\alpha}(\tilde{J}_{n+1}-\tilde{J}_n)\right\rangle$$

$$
\begin{aligned}
&= \frac{\rho}{\Delta t}\frac{1}{(\alpha-\frac{1}{2})}\left[\left\langle \tilde{\boldsymbol{v}}_{n+\alpha}, (\tilde{\boldsymbol{v}}_{n+\alpha}-\tilde{\boldsymbol{v}}_{n+\frac{1}{2}})\tilde{J}_{n+\alpha}\right\rangle + \frac{1}{2}\left\langle \tilde{\boldsymbol{v}}_{n+\alpha}, \tilde{\boldsymbol{v}}_{n+\alpha}(\tilde{J}_{n+\alpha}-\tilde{J}_{n+\frac{1}{2}})\right\rangle\right] \\
&= \frac{\rho}{\Delta t}\frac{1}{(\alpha-\frac{1}{2})}\left[\frac{3}{2}\left\langle \tilde{\boldsymbol{v}}_{n+\alpha}, \tilde{\boldsymbol{v}}_{n+\alpha}\tilde{J}_{n+\alpha}\right\rangle - \left\langle \tilde{\boldsymbol{v}}_{n+\alpha}, \tilde{\boldsymbol{v}}_{n+\frac{1}{2}}\tilde{J}_{n+\alpha}\right\rangle - \frac{1}{2}\left\langle \tilde{\boldsymbol{v}}_{n+\alpha}, \tilde{\boldsymbol{v}}_{n+\alpha}\tilde{J}_{n+\frac{1}{2}}\right\rangle\right] \\
&= \frac{\rho}{\Delta t}\frac{1}{(\alpha-\frac{1}{2})}\left[\frac{3}{2}\left\langle \alpha\tilde{\boldsymbol{v}}_{n+1}+(1-\alpha)\,\tilde{\boldsymbol{v}}_n, \alpha\tilde{\boldsymbol{v}}_{n+1}+(1-\alpha)\,\tilde{\boldsymbol{v}}_n\alpha\tilde{J}_{n+1}+(1-\alpha)\,\tilde{J}_n\right\rangle\right. \\
&\qquad\qquad - \left\langle \alpha\tilde{\boldsymbol{v}}_{n+1}+(1-\alpha)\,\tilde{\boldsymbol{v}}_n, \frac{1}{2}\left(\tilde{\boldsymbol{v}}_{n+1}+\tilde{\boldsymbol{v}}_n\right)\alpha\tilde{J}_{n+1}+(1-\alpha)\,\tilde{J}_n\right\rangle \\
&\qquad\qquad \left. - \frac{1}{2}\left\langle \alpha\tilde{\boldsymbol{v}}_{n+1}+(1-\alpha)\,\tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{v}}_{n+\alpha}\frac{1}{2}\left(\tilde{J}_{n+1}+\tilde{J}_n\right)\right\rangle\right] \\
&= 3\alpha^2\frac{1}{\Delta t}\left(\tilde{K}(\tilde{\boldsymbol{v}}_{n+1})-\tilde{K}(\tilde{\boldsymbol{v}}_n)\right) + \frac{1}{\Delta t}6\left(\alpha-\frac{1}{2}\right)\tilde{K}(\tilde{\boldsymbol{v}}_n) \\
&\quad + \frac{\rho}{\Delta t}\left[\frac{\alpha}{2}(2-3\alpha)\left\langle \tilde{\boldsymbol{v}}_{n+1}, \tilde{\boldsymbol{v}}_{n+1}J_n\right\rangle + \alpha(2-3\alpha)\left\langle \tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{v}}_{n+1}J_{n+1}\right\rangle\right. \\
&\quad \left. - \frac{1}{4}(1-\alpha)(3\alpha-1)\left\langle \tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{v}}_{n+1}J_n\right\rangle + \frac{(1-\alpha)^2(6\alpha-1)}{4(\alpha-\frac{1}{2})}\left\langle \tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{v}}_nJ_{n+1}\right\rangle\right]
\end{aligned}
$$

(by repeated substitution of (14) and (16)). Thus the ramifications of Assumption 1 are that the total

$$
\begin{aligned}
&\frac{1}{\Delta t}6\left(\alpha-\frac{1}{2}\right)\tilde{K}(\tilde{\boldsymbol{v}}_n) + \frac{\rho}{\Delta t}\left[\frac{\alpha}{2}(2-3\alpha)\left\langle \tilde{\boldsymbol{v}}_{n+1}, \tilde{\boldsymbol{v}}_{n+1}J_n\right\rangle + \alpha(2-3\alpha)\left\langle \tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{v}}_{n+1}J_{n+1}\right\rangle\right. \\
&\qquad\qquad \left. - \frac{1}{4}(1-\alpha)(3\alpha-1)\left\langle \tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{v}}_{n+1}J_n\right\rangle + \frac{(1-\alpha)^2(6\alpha-1)}{4(\alpha-\frac{1}{2})}\left\langle \tilde{\boldsymbol{v}}_n, \tilde{\boldsymbol{v}}_nJ_{n+1}\right\rangle\right]
\end{aligned}
$$

is positive, furthermore it is sufficiently positive to offset any subsequent short–coming which arises when Assumption 1 is exploited in the proof of Lemma 7 i.e. in the event of

$$4\nu C\alpha(1-\alpha)\left(\left|\left|\tilde{\boldsymbol{v}}_{n+1}\tilde{J}_{n+1}\right|\right|^2 - \left|\left|\tilde{\boldsymbol{v}}_{n+1}\tilde{J}_{n+\alpha}\right|\right|^2\right)$$

and

$$4\nu C\alpha(1-\alpha)\left(\left|\left|\tilde{\boldsymbol{v}}_n\tilde{J}_n\right|\right|^2 - \left|\left|\tilde{\boldsymbol{v}}_n\tilde{J}_{n+\alpha}\right|\right|^2\right)$$

not being positive.

# 9 Appendix II (Proofs)

### Proof of Relation 1

The above relation (taken from Hughes, Liu and Zimmerman [6]) is obtained by recalling that the material derivative (total derivative) is the derivative with respect to time in the material configuration. Thus

$$\frac{D\tilde{v}_i}{Dt} = \frac{\partial}{\partial t}\{\tilde{v}_i(\tilde{\boldsymbol{\lambda}}(\boldsymbol{x}_0,t),t)\}$$

$$= \frac{\partial \tilde{v}_i}{\partial t} + \frac{\partial \tilde{v}_i}{\partial \tilde{x}_j}\frac{\partial \tilde{\lambda}_j}{\partial t} . \tag{22}$$

A more practical expression is needed for $\frac{\partial \tilde{\lambda}_j}{\partial t}$ (the velocity as perceived in the distorting reference). This can be obtained by considering

$$\lambda_k(\boldsymbol{x}_0, t) = \lambda_k^*(\tilde{\boldsymbol{\lambda}}(\boldsymbol{x}_0, t), t) \qquad \text{(see Fig. 1)}$$

so that

$$\left.\frac{\partial \lambda_k}{\partial t}\right|_{\boldsymbol{x}_0 \ fixed} = \left.\frac{\partial \lambda_k^*}{\partial t}\right|_{\tilde{\boldsymbol{x}} \ fixed} + \frac{\partial \lambda_k^*}{\partial \tilde{x}_j}\frac{\partial \tilde{\lambda}_j}{\partial t}$$

or

$$\frac{\partial \tilde{\lambda}_j}{\partial t} = \frac{\partial \tilde{x}_j}{\partial x_k}\left(\left.\frac{\partial \lambda_k}{\partial t}\right|_{\boldsymbol{x}_0 \ fixed} - \left.\frac{\partial \lambda_k^*}{\partial t}\right|_{\tilde{\boldsymbol{x}} \ fixed}\right).$$

Substituting this expression into equation (22), the desired, suitably practicable result is obtained.

## Proof of Inequality 1

Consider the change to spherical coordinates

$$\breve{v}_i(r, \theta, \phi) = v_i(r \sin\theta\cos\phi - x_1^{\text{origin}}, r\sin\theta\sin\phi - x_2^{\text{origin}}, r\cos\theta - x_3^{\text{origin}})$$

centred on $\boldsymbol{x}^{\text{origin}}$. Suppose the radial limits of the domain and neighbourhood are denoted $R_b(\theta, \phi)$ and $R_a(\theta, \phi)$ respectively. By the fundamental theorem of integral calculus

$$\begin{aligned}
\left(\breve{v}_i(r,\theta,\phi) - \breve{v}_i\,|_{R_a(\theta,\phi)}\right)^2 &= \left(\int_{R_a(\theta,\phi)}^{r} \frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi)d\xi\right)^2 \\
&= \left(\int_{R_a(\theta,\phi)}^{r} \frac{1}{\xi}\xi\frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi)d\xi\right)^2 \\
&\leq \int_{R_a(\theta,\phi)}^{r} \frac{1}{\xi^2}d\xi \int_{R_a(\theta,\phi)}^{r}\left(\frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi)\right)^2 \xi^2 d\xi \\
&\qquad \textit{(by Schwarz inequality)} \\
&\leq \int_{R_{\min}}^{R_{\max}} \frac{1}{\xi^2}d\xi \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)}\left(\frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi)\right)^2 \xi^2 d\xi \qquad \textit{(for } r \in \breve{\Omega}\textit{)} \\
&= \frac{(R_{\max} - R_{\min})}{R_{\max}R_{\min}} \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)}\left(\frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi)\right)^2 \xi^2 d\xi \\
&= \frac{(R_{\max} - R_{\min})}{R_{\max}R_{\min}}\breve{V}_i(\theta,\phi)
\end{aligned}$$

$$\text{where} \qquad \breve{V}_i(\theta,\phi) = \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)}\left(\frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi)\right)^2 \xi^2 d\xi.$$

Integrating this result over that part of $\breve{\Omega}$ outside the neighbourhood (angular extent being $\Theta_a(\phi) \leq \theta \leq \Theta_b(\phi)$ and $\Phi_a \leq \phi \leq \Phi_b$)

$$
\begin{aligned}
\int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} & \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} \left( \breve{v}_i(r,\theta,\phi) - \breve{v}_i \mid_{R_a(\theta,\phi)} \right)^2 r^2 \sin\theta dr d\theta d\phi \\
&\leq \frac{(R_{\max} - R_{\min})}{R_{\max} R_{\min}} \int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} \breve{V}_i(\theta,\phi) r^2 \sin\theta dr d\theta d\phi \\
&\leq \frac{(R_{\max} - R_{\min})}{R_{\max} R_{\min}} \int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} \breve{V}_i(\theta,\phi) \left( \int_{R_{\min}}^{R_{\max}} r^2 dr \right) \sin\theta d\theta d\phi \\
&\leq \frac{(R_{\max} - R_{\min})(R_{\max}^3 - R_{\min}^3)}{3 R_{\max} R_{\min}} \int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} \left( \frac{\partial \breve{v}_i}{\partial r} \right)^2 r^2 \sin\theta dr d\theta d\phi \\
&\leq \frac{(R_{\max} - R_{\min})(R_{\max}^3 - R_{\min}^3)}{3 R_{\max} R_{\min}} \int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} \left[ \left( \frac{\partial \breve{v}_i}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial \breve{v}_i}{\partial \theta} \right)^2 \right. \\
&\qquad\qquad \left. + \frac{1}{r^2 \sin^2\theta} \left( \frac{\partial \breve{v}_i}{\partial \phi} \right)^2 \right] r^2 \sin\theta dr d\theta d\phi \\
&= \frac{(R_{\max} - R_{\min})(R_{\max}^3 - R_{\min}^3)}{3 R_{\max} R_{\min}} \int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} (\nabla \breve{v}_i) \cdot (\nabla \breve{v}_i) \, r^2 \sin\theta dr d\theta d\phi .
\end{aligned}
$$

Changing back to the original rectangular coordinates and defining $\boldsymbol{v} \mid_{\text{bndry}}$ to be a radially constant function throughout $\Omega$ which takes the values of $\breve{\boldsymbol{v}} \mid_{R_a(\theta,\phi)}$ for $r = R_a(\theta,\phi)$,

$$
\int_{\Omega_*} \left( v_i(\boldsymbol{x}) - v_i \mid_{\text{bndry}} \right)^2 d\Omega \quad \leq \quad \frac{(R_{\max} - R_{\min})(R_{\max}^3 - R_{\min}^3)}{3 R_{\max} R_{\min}} \int_{\Omega_*} (\nabla v_i(\boldsymbol{x})) \cdot (\nabla v_i(\boldsymbol{x})) \, d\Omega
$$

where $\Omega_*$ is $\Omega$ excluding the neighbourhood. Summing over $i$,

$$
\int_{\Omega_*} (\boldsymbol{v} - \boldsymbol{v} \mid_{\text{bndry}}) \cdot (\boldsymbol{v} - \boldsymbol{v} \mid_{\text{bndry}}) \, d\Omega \quad \leq \quad \frac{(R_{\max} - R_{\min})(R_{\max}^3 - R_{\min}^3)}{3 R_{\max} R_{\min}} \int_{\Omega_*} (\nabla \boldsymbol{v}) : (\nabla \boldsymbol{v}) \, d\Omega .
$$

Making use of either the Cauchy–Schwarz or triangle inequality,

$$
\left( ||\boldsymbol{v}||_{L^2(\Omega_*)} - ||\boldsymbol{v} \mid_{\text{bndry}}||_{L^2(\Omega_*)} \right)^2 \quad \leq \quad \frac{(R_{\max} - R_{\min})(R_{\max}^3 - R_{\min}^3)}{3 R_{\max} R_{\min}} \, ||\nabla \boldsymbol{v}||^2_{L^2(\Omega_*)} ,
$$

and remembering that $\sup \mid \breve{\boldsymbol{v}} \mid_{R_a(\theta,\phi)} \mid \leq c$,

$$
||\boldsymbol{v}||_{L^2(\Omega_*)} \quad \leq \quad \left[ \frac{(R_{\max} - R_{\min})(R_{\max}^3 - R_{\min}^3)}{3 R_{\max} R_{\min}} \right]^{\frac{1}{2}} ||\nabla \boldsymbol{v}||_{L^2(\Omega_*)} + ||c||_{L^2(\Omega_*)} .
$$

Consider the terms $||\boldsymbol{v}||_{L^2}$ and $||c||_{L^2}$. Comparing these terms under circumstances of $\sup \mid \boldsymbol{v} \mid \leq c$ leads to the conclusion that the inequality holds over the neighbourhood and that the inequality is therefore unaffected when the domain of integration is extended to include the neighbourhood. Of course, the radial extension of $\boldsymbol{v} \mid_{\text{bndry}}$ can be used in place of $c$ in instances where inclusion of the neighbourhood is not required.

**Proof of Lemma 1**

If, in particular, $\boldsymbol{v}\,|_{\text{bndry}}=0$ in Inequality 1,

$$\begin{aligned} ||\boldsymbol{v}||_{L^2(\Omega)} &\leq \frac{||\nabla \boldsymbol{v}||_{L^2(\Omega)}}{\sqrt{C}} \\ C\frac{1}{2}\,||\boldsymbol{v}||^2_{L^2(\Omega)} &\leq ||\boldsymbol{D}(\boldsymbol{v})||^2_{L^2(\Omega)} \end{aligned}$$

(The relationship between $\boldsymbol{D}$ and $\nabla \boldsymbol{v}$ arises in the context of the original equations involving $\operatorname{div}\boldsymbol{\sigma}$. It is because

$$\begin{aligned} D_{ij,j} &= \tfrac{1}{2}\left(v_{i,jj}+v_{j,ij}\right) && \\ &= \tfrac{1}{2}\left(v_{i,jj}+v_{j,ji}\right) && \textit{(changing the order of differentiation)} \\ &= \tfrac{1}{2}v_{i,jj} && (\operatorname{div}\boldsymbol{v}=0 \textit{ by incompressibility}), \end{aligned}$$

assuming, of course, that $\boldsymbol{v}$ is continuous and differentiable to first order.) Rewriting in terms of $\tilde{\Omega}$

$$\frac{C}{\rho}\tilde{K}(\tilde{\boldsymbol{v}}) \leq \left|\left|\tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}})\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}.$$

**Proof of Lemma 2**

Consider $\tilde{\boldsymbol{u}}\cdot(\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\tilde{\boldsymbol{w}}\tilde{J}$:

$$\tilde{u}_i\tilde{v}_{i,j}\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J} = -\tilde{u}_{i,j}\tilde{v}_i\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J} - \tilde{u}_i\tilde{v}_i(\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J})_{,j} + (\tilde{u}_i\tilde{v}_i\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J})_{,j}$$

by the product rule. In the terms arising from $(\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J})_{,j}$, both $\tilde{F}^{-1}_{jk,j}$ and $\tilde{J}_{,j}\tilde{F}^{-1}_{jk}$ vanish under the condtions specified (in section 2.4) for equations (10) and (11) to be a completely general reference description. Thus

$$\tilde{u}_i\tilde{v}_{i,j}\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J} = -\tilde{u}_{i,j}\tilde{v}_i\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J} - \tilde{u}_i\tilde{v}_i\tilde{F}^{-1}_{jk}\tilde{w}_{k,j}\tilde{J} + (\tilde{u}_i\tilde{v}_i\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J})_{,j}.$$

Integrating over the domain $\tilde{\Omega}$ and applying the divergence theorem,

$$\begin{aligned} \left\langle \tilde{\boldsymbol{u}}, (\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\tilde{\boldsymbol{w}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} &= -\left\langle \tilde{\boldsymbol{v}}, (\tilde{\nabla}\tilde{\boldsymbol{u}})\tilde{\boldsymbol{F}}^{-1}\tilde{\boldsymbol{w}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \\ &\quad -\left\langle \tilde{\boldsymbol{u}}\left(\tilde{\nabla}\tilde{\boldsymbol{w}} : \tilde{\boldsymbol{F}}^{-t}\right), \tilde{\boldsymbol{v}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \\ &\quad +\left\langle \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}\left(\tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot\tilde{\boldsymbol{w}}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Gamma})}. \end{aligned} \tag{23}$$

Thus the term

$$\begin{aligned} 2\left\langle \tilde{\boldsymbol{v}}, (\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} &= -\left\langle \tilde{\boldsymbol{v}}\left(\tilde{\nabla}\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right) : \tilde{\boldsymbol{F}}^{-t}\right), \tilde{\boldsymbol{v}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \\ &\quad +\left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}\left(\tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref})\right)\tilde{J}\right\rangle_{L^2(\tilde{\Gamma})} \end{aligned}$$

$$
\begin{aligned}
&= \left\langle \tilde{\boldsymbol{v}} \left( \tilde{\nabla}\tilde{\boldsymbol{v}}^{ref} : \tilde{\boldsymbol{F}}^{-t} \right), \tilde{\boldsymbol{v}}\tilde{J} \right\rangle_{L^2(\tilde{\Omega})} \\
&\quad + \left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}} \left( \tilde{\boldsymbol{F}}^{-t} \tilde{\boldsymbol{N}} \cdot (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}) \right) \tilde{J} \right\rangle_{L^2(\tilde{\Gamma})} \\
&\qquad \textit{(by incompressibility)} \\
&= \left\langle \tilde{\boldsymbol{v}} \left( \tilde{\nabla}\tilde{\boldsymbol{v}}^{ref} : \tilde{\boldsymbol{F}}^{-t} \right), \tilde{\boldsymbol{v}}\tilde{J} \right\rangle_{L^2(\tilde{\Omega})} \\
&\quad + \left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}} \left( \tilde{\boldsymbol{F}}^{-t} \tilde{\boldsymbol{N}} \cdot (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}) \right) \tilde{J} \right\rangle_{L^2(\tilde{\Gamma})} \\
&= \left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}} \frac{\partial \tilde{J}}{\partial t} \right\rangle_{L^2(\tilde{\Omega})} \\
&\quad + \left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}} \left( \tilde{\boldsymbol{F}}^{-t} \tilde{\boldsymbol{N}} \cdot (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}) \right) \tilde{J} \right\rangle_{L^2(\tilde{\Gamma})}
\end{aligned}
$$

since $\dfrac{\partial \tilde{J}}{\partial t} = \tilde{J} div \boldsymbol{v}^{ref}$ (which is $\tilde{J}\tilde{\nabla}\tilde{\boldsymbol{v}}^{ref} : \tilde{\boldsymbol{F}}^{-t}$) in the same vein as $\dot{\mathcal{J}}_0 = \mathcal{J}_0 \operatorname{div} \boldsymbol{v}$ (the kinematic result used in Section 2).

Equation (23) is vital to the deforming reference analysis in particular. It forms the basis to this lemma and another (Lemma 6) concerned with the time discrete analysis.

## Proof of Lemma 3

By referring to Fig. 1 one observes that

$$\lambda_j(\boldsymbol{x}_0, t) = \lambda_j^*(\tilde{\boldsymbol{\lambda}}(\boldsymbol{x}_0, t), t)$$

and consequently that

$$\left. \frac{\partial \lambda_j}{\partial t} \right|_{\boldsymbol{x}_0 \ fixed} = \left. \frac{\partial \lambda_j^*}{\partial t} \right|_{\tilde{\boldsymbol{x}} \ fixed} + \frac{\partial \lambda_j^*}{\partial \tilde{x}_i} \frac{\partial \tilde{\lambda}_i}{\partial t}.$$

That is

$$\tilde{\boldsymbol{F}}^{-1}(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}) = \frac{\partial \tilde{\boldsymbol{\lambda}}}{\partial t}.$$

In other words $\tilde{\boldsymbol{F}}^{-1}(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref})$ is the velocity perceived in the deforming reference. This perceived velocity is tangent to the free surface since the description was stipulated to be one in which $\tilde{\boldsymbol{n}} \cdot \left( \tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref} \right)$ vanishes at free surfaces. Remembering that $\tilde{\boldsymbol{N}}$ is a surface normal as defined in terms of this self–same reference,

$$\tilde{\boldsymbol{N}} \cdot \tilde{\boldsymbol{F}}^{-1} \left( \tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref} \right) = 0 \quad \Rightarrow \quad -\frac{1}{2}\rho \left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}} \left( \tilde{\boldsymbol{F}}^{-t} \tilde{\boldsymbol{N}} \cdot \left( \tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref} \right) \right) \tilde{J} \right\rangle_{L^2(\tilde{\Gamma})} = 0$$

at free boundaries.

**Proof of Lemma 4**

In terms of the Cauchy–Schwarz inequality,

$$\begin{aligned}\left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{b}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} &\leq \left|\left|\tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}}\right|\right|_{L^2(\tilde{\Omega})} \left|\left|\tilde{\boldsymbol{b}}\tilde{J}^{\frac{1}{2}}\right|\right|_{L^2(\tilde{\Omega})} \\ &\leq \frac{\nu C}{2}\left|\left|\tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})} + \frac{1}{2\nu C}\left|\left|\tilde{\boldsymbol{b}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})} \quad \text{for} \quad \nu C > 0\end{aligned}$$

by Young's inequality. Similarly,

$$\left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right\rangle_{L^2(\tilde{\Gamma})} \leq \frac{\nu C}{2}\left|\left|\tilde{\boldsymbol{v}}\right|\right|^2_{L^2(\tilde{\Gamma})} + \frac{1}{2\nu C}\left|\left|\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right|\right|^2_{L^2(\tilde{\Gamma})} \quad \text{for} \quad \nu C > 0.$$

**Proof of Lemma 5**

For a description which becomes purely Eulerian at a fixed boundary across which there is an imposed velocity,

$$\begin{aligned}-\frac{1}{2}\rho\left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}\left(\tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)\right)\tilde{J}\right\rangle_{L^2(\tilde{\Gamma})} &= -\frac{1}{2}\rho\left\langle \boldsymbol{v}, \boldsymbol{v}\left(\boldsymbol{n}\cdot\boldsymbol{v}\right)\right\rangle_{L^2(\Gamma)} \\ &\leq \frac{1}{2}\rho\left|\int_\Gamma\left[\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{n}\cdot\boldsymbol{v})^{\frac{1}{2}}\right](\boldsymbol{n}\cdot\boldsymbol{v})^{\frac{1}{2}}\, d\Gamma\right| \\ &\leq \frac{1}{2}\rho\left(\int_\Gamma \boldsymbol{n}\cdot\boldsymbol{v}\, d\Gamma\right)^{\frac{1}{2}}\left(\int_\Gamma(\boldsymbol{v}\cdot\boldsymbol{v})^2\,\boldsymbol{n}\cdot\boldsymbol{v}\, d\Gamma\right)^{\frac{1}{2}} \\ &\qquad \textit{(by Schwarz inequality)} \\ &= 0\end{aligned}$$

since the total flow across the boundaries, $\int_\Gamma \boldsymbol{n}\cdot\boldsymbol{v}\, d\Gamma$ of a fixed volume of incompressible fluid must be zero.

**Proof of Lemma 6**

By equation (23) of the Lemma 2 proof.

**Proof of Lemma 7**

By Young's inequality

$$\begin{aligned}\left|\left|\tilde{\boldsymbol{v}}_{n+1}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right|\right|_{L^2(\tilde{\Omega}_{n+\alpha})}\left|\left|\tilde{\boldsymbol{v}}_{n}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right|\right|_{L^2(\tilde{\Omega}_{n+\alpha})} &\leq \left(\frac{c}{2}\right)\left|\left|\tilde{\boldsymbol{v}}_{n+1}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})} \\ &\quad + \left(\frac{1}{2c}\right)\left|\left|\tilde{\boldsymbol{v}}_{n}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})} \qquad (24)\end{aligned}$$

for $c > 0$. Writing $\tilde{K}(\tilde{\boldsymbol{v}}_{n+\alpha})$ explicitly in terms of the "intermediate" velocity definition, (14), leads to

$$\begin{aligned}
\tilde{K}(\tilde{\boldsymbol{v}}_{n+\alpha}) &= \alpha^2 \tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) + (1-\alpha)^2 \tilde{K}(\tilde{\boldsymbol{v}}_n) + 2\alpha(1-\alpha) \left\langle \tilde{\boldsymbol{v}}_{n+1}, \tilde{\boldsymbol{v}}_n \tilde{J}_{n+\alpha} \right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} \\
&\geq \alpha^2 \tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) + (1-\alpha)^2 \tilde{K}(\tilde{\boldsymbol{v}}_n) \\
&\quad -2\alpha(1-\alpha) \left\| \tilde{\boldsymbol{v}}_{n+1} \tilde{J}^{\frac{1}{2}}_{n+\alpha} \right\|_{L^2(\tilde{\Omega}_{n+\alpha})} \left\| \tilde{\boldsymbol{v}}_n \tilde{J}^{\frac{1}{2}}_{n+\alpha} \right\|_{L^2(\tilde{\Omega}_{n+\alpha})} \\
&\geq \alpha \left[\alpha - (1-\alpha)c\right] \tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) + (1-\alpha) \left[(1-\alpha) - \frac{\alpha}{c}\right] \tilde{K}(\tilde{\boldsymbol{v}}_n)
\end{aligned}$$

using equation (24).

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